Pre-Calculus Honors Summer Assignment

Your summer assignment is to review topics covered in Algebra II. For this assignment, you will need a one subject notebook, scientific calculator and Pre-Calculus textbook. You must complete each assigned problem in the notebook. You must show your work for all problems and circle your final answer. Answers and work should be neat and legible. Anything I cannot read will be marked incorrect. Your work must match your answer**. The notebook will be collected on the first Friday of school.**

The summer assignment will be your first test grade of MP 1. If the notebook is turned in one day late; 30 points will be taken off the test grade, 2 days late; 60 points will be taken off, 3 days late; you will receive a 0 for the assignment. Keep in mind that tests are worth 60% of your marking period grade. Please make sure you do the correct questions. If you do the wrong problems, those will not be graded and the questions will be marked incorrect.

YOU ARE TO COMPLETE THE FOLLOWING:

A.1 Review material on Algebra Essentials #'s:

24, 28, 34, 40, 48, 52, 56, 58, 60, 68, 74, 78, 82, 86, 94, 96, 102, 106, 114, 124, 138, 140

A.2 Review material on Geometry Essentials #'s:

12, 16, 22, 28, 30, 32, 38, 42, 48

A.3 Review material on Polynomials #'s:

14, 18, 24, 30, 34, 38, 46, 52, 58, 64, 72, 74, 80, 90, 98, 108, 118, 132

A.4 Review material on Synthetic Division #'s:

8, 16, 20, 26

A.5 Review material on Rational Expressions #'s:

8, 10, 16, 20, 28, 32

A.6 Review material on Solving Equations #'s:

16, 20, 24, 34, 38, 42, 50, 60, 68, 74, 76, 82, 86, 90, 96, 104, 110, 114

A.7 Review material on Complex Numbers #'s:

10, 16, 20, 30, 38, 48, 52, 58, 64, 70, 78

A.8 Review material on Problem Solving #'s:

8, 10, 12, 14, 16, 20, 26, 32, 36, 40, 50, 54 A.9

Review material on Interval Notation #'s:

12, 18, 22, 28, 38, 42, 52, 56, 64, 72, 78, 90, 98

A.10 Review material on Nth Roots #'s:

8, 16, 26, 36, 38, 42, 50, 56, 68, 78, 90, 104

You may contact me via e-mail with questions: kbrennan2@hopatcongschools.org; however, I do not expect there to be any major questions. If you do not hear back from me within a week, e-mail me again! There are a total of 125 problems for you to complete this summer. DO NOT wait until the last minute to start/complete the assignment. NO WORK = NO CREDIT! Good Luck and Enjoy Your Summer!!!

Appendix A
Review

Outline

A.1 Algebra Essentials

PREPARING FOR THIS SECTION *Before getting started, read "To the Student" on page ii at the beginning of this book.*

OBJECTIVES
1 Work with Sets (p. A1)
2 Graph Inequalities (p. A4)
3 Find Distance on the Real Number Line (p. A5)
4 Evaluate Algebraic Expressions (p. A6)
5 Determine the Domain of a Variable (p. A7)
6 Use the Laws of Exponents (p. A7)
7 Evaluate Square Roots (p. A9)
8 Use a Calculator to Evaluate Exponents (p. A10)

1 Work with Sets

A **set** is a well-defined collection of distinct objects. The objects of a set are called its **elements.** By **well-defined,** we mean that there is a rule that enables us to determine whether a given object is an element of the set. If a set has no elements, it is called the **empty set,** or **null set,** and is denoted by the symbol $\varnothing$.

For example, the set of *digits* consists of the collection of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. If we use the symbol D to denote the set of digits, then we can write

$$D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

In this notation, the braces { } are used to enclose the objects, or **elements,** in the set. This method of denoting a set is called the **roster method.** A second way to denote a set is to use **set-builder notation,** where the set D of digits is written as

$$D = \{\ x \mid x \text{ is a digit}\}$$

Read as "D is the set of all x such that x is a digit."

EXAMPLE 1 **Using Set-builder Notation and the Roster Method**

(a) $E = \{x|x \text{ is an even digit}\} = \{0, 2, 4, 6, 8\}$
(b) $O = \{x|x \text{ is an odd digit}\} = \{1, 3, 5, 7, 9\}$

Because the elements of a set are distinct, we never repeat elements. For example, we would never write $\{1, 2, 3, 2\}$; the correct listing is $\{1, 2, 3\}$. Because a set is a collection, the order in which the elements are listed is immaterial. $\{1, 2, 3\}$, $\{1, 3, 2\}$, $\{2, 1, 3\}$, and so on, all represent the same set.

If every element of a set A is also an element of a set B, then we say that A is a **subset** of B and write $A \subseteq B$. If two sets A and B have the same elements, then we say that A **equals** B and write $A = B$.

For example, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ and $\{1, 2, 3\} = \{2, 3, 1\}$.

DEFINITION If A and B are sets, the **intersection** of A with B, denoted $A \cap B$, is the set consisting of elements that belong to both A and B. The **union** of A with B, denoted $A \cup B$, is the set consisting of elements that belong to either A or B, or both.

EXAMPLE 2 **Finding the Intersection and Union of Sets**

Let $A = \{1, 3, 5, 8\}$, $B = \{3, 5, 7\}$, and $C = \{2, 4, 6, 8\}$. Find:

(a) $A \cap B$ (b) $A \cup B$ (c) $B \cap (A \cup C)$

Solution

(a) $A \cap B = \{1, 3, 5, 8\} \cap \{3, 5, 7\} = \{3, 5\}$
(b) $A \cup B = \{1, 3, 5, 8\} \cup \{3, 5, 7\} = \{1, 3, 5, 7, 8\}$
(c) $B \cap (A \cup C) = \{3, 5, 7\} \cap (\{1, 3, 5, 8\} \cup \{2, 4, 6, 8\})$
$= \{3, 5, 7\} \cap \{1, 2, 3, 4, 5, 6, 8\} = \{3, 5\}$

Now Work PROBLEM 13

Usually, in working with sets, we designate a **universal set** U, the set consisting of all the elements that we wish to consider. Once a universal set has been designated, we can consider elements of the universal set not found in a given set.

DEFINITION If A is a set, the **complement** of A, denoted $\overline{A}$, is the set consisting of all the elements in the universal set that are not in A.*

EXAMPLE 3 **Finding the Complement of a Set**

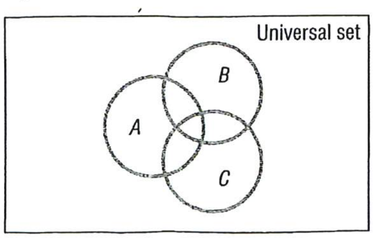
If the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and if $A = \{1, 3, 5, 7, 9\}$, then $\overline{A} = \{2, 4, 6, 8\}$.

It follows from the definition of complement that $A \cup \overline{A} = U$ and $A \cap \overline{A} = \varnothing$. Do you see why?

Now Work PROBLEM 17

It is often helpful to draw pictures of sets. Such pictures, called **Venn diagrams,** represent sets as circles enclosed in a rectangle, which represents the universal set. Such diagrams often help us to visualize various relationships among sets. See Figure 1.

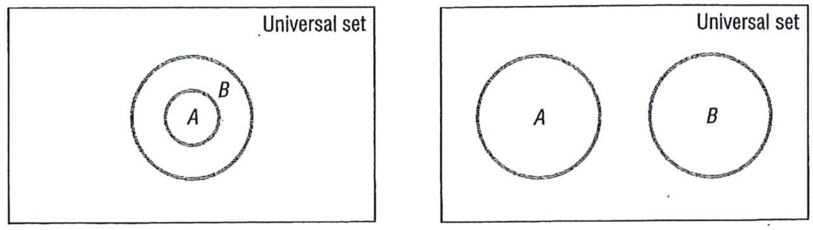
Figure 1



* Some books use the notation A' or A^c for the complement of A.

If we know that $A \subseteq B$, we might use the Venn diagram in Figure 2(a). If we know that A and B have no elements in common, that is, if $A \cap B = \varnothing$, we might use the Venn diagram in Figure 2(b). The sets A and B in Figure 2(b) are said to be **disjoint.**

Figure 2

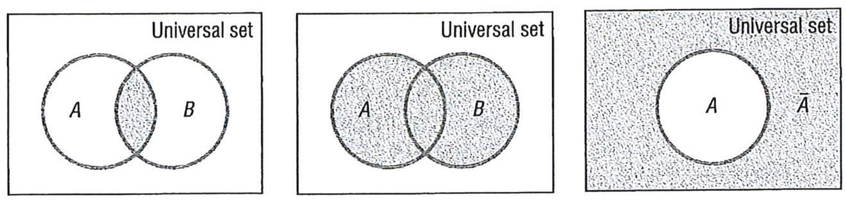


(a) $A \subseteq B$ subset

(b) $A \cap B = \varnothing$ disjoint sets

Figures 3(a), 3(b), and 3(c) use Venn diagrams to illustrate the definitions of intersection, union, and complement, respectively.

Figure 3



(a) $A \cap B$ intersection

(b) $A \cup B$ union

(c) $\bar{A}$ complement

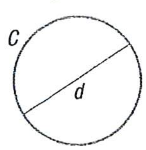
Real Numbers

Real numbers are represented by symbols such as

$$25, \quad 0, \quad -3, \quad \frac{1}{2}, \quad -\frac{5}{4}, \quad 0.125, \quad \sqrt{2}, \quad \pi, \quad \sqrt[3]{-2}, \quad 0.666\ldots$$

The set of **counting numbers,** or **natural numbers,** contains the numbers in the set $\{1, 2, 3, 4, \ldots\}$. (The three dots, called an **ellipsis,** indicate that the pattern continues indefinitely.) The set of **integers** contains the numbers in the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. A **rational number** is a number that can be expressed as a *quotient* $\frac{a}{b}$ of two integers, where the integer b cannot be 0. Examples of rational numbers are $\frac{3}{4}, \frac{5}{2}, \frac{0}{4}$, and $-\frac{2}{3}$. Since $\frac{a}{1} = a$ for any integer a, every integer is also a rational number. Real numbers that are not rational are called **irrational.** Examples of irrational numbers are $\sqrt{2}$ and π (the Greek letter pi), which equals the constant ratio of the circumference to the diameter of a circle. See Figure 4.

Figure 4 $\pi = \frac{C}{d}$



Real numbers can be represented as **decimals.** Rational real numbers have decimal representations that either **terminate** or are nonterminating with **repeating** blocks of digits. For example, $\frac{3}{4} = 0.75$, which terminates; and $\frac{2}{3} = 0.666\ldots$, in which the digit 6 repeats indefinitely. Irrational real numbers have decimal representations that neither repeat nor terminate. For example, $\sqrt{2} = 1.414213\ldots$ and $\pi = 3.14159\ldots$. In practice, the decimal representation of an irrational number is given as an approximation. We use the symbol $\approx$ (read as "approximately equal to") to write $\sqrt{2} \approx 1.4142$ and $\pi \approx 3.1416$.

Two properties of real numbers that we shall use often are given next. Suppose that a, b, and c are real numbers.

Distributive Property

$$a \cdot (b + c) = ab + ac$$

In Words
If a product equals 0, then one or both of the factors is 0.

Zero-Product Property

If $ab = 0$, then either $a = 0$ or $b = 0$ or both equal 0.

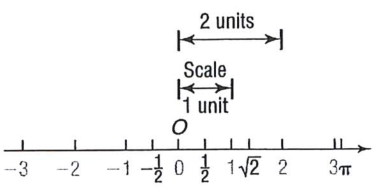
The Distributive Property can be used to remove parentheses:

$$2(x + 3) = 2x + 2 \cdot 3 = 2x + 6$$

The Zero-Product Property will be used to solve equations (Section A.6). For example, if $2x = 0$, then $2 = 0$ or $x = 0$. Since $2 \neq 0$, it follows that $x = 0$.

The Real Number Line

Figure 5
Real number line



The real numbers can be represented by points on a line called the **real number line.** There is a one-to-one correspondence between real numbers and points on a line. That is, every real number corresponds to a point on the line, and each point on the line has a unique real number associated with it.

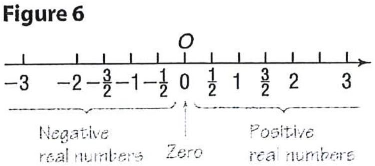
Pick a point on the line somewhere in the center, and label it O. This point, called the **origin,** corresponds to the real number 0. See Figure 5. The point 1 unit to the right of O corresponds to the number 1. The distance between 0 and 1 determines the **scale** of the number line. For example, the point associated with the number 2 is twice as far from O as 1. Notice that an arrowhead on the right end of the line indicates the direction in which the numbers increase. Points to the left of the origin correspond to the real numbers -1, -2, and so on. Figure 5 also shows the points associated with the rational numbers $-\frac{1}{2}$ and $\frac{1}{2}$ and with the irrational numbers $\sqrt{2}$ and π.

DEFINITION

The real number associated with a point P is called the **coordinate** of P, and the line whose points have been assigned coordinates is called the **real number line.**

The real number line consists of three classes of real numbers, as shown in Figure 6.

Figure 6

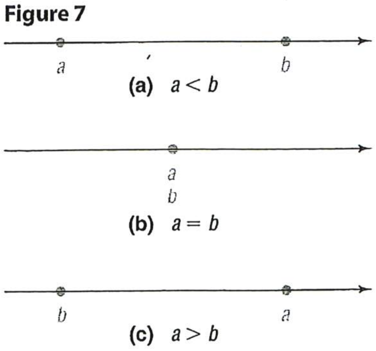


1. The **negative real numbers** are the coordinates of points to the left of the origin O.
2. The real number **zero** is the coordinate of the origin O.
3. The **positive real numbers** are the coordinates of points to the right of the origin O.

Now Work PROBLEM 21

2 Graph Inequalities

Figure 7



An important property of the real number line follows from the fact that, given two numbers (points) a and b, either a is to the left of b, or a is at the same location as b, or a is to the right of b. See Figure 7.

If a is to the left of b, we say that "a is less than b" and write $a < b$. If a is to the right of b, we say that "a is greater than b" and write $a > b$. If a is at the same location as b, then $a = b$. If a is either less than or equal to b, we write $a \leq b$. Similarly, $a \geq b$ means that a is either greater than or equal to b. Collectively, the symbols $<$, $>$, $\leq$, and $\geq$ are called **inequality symbols.**

Note that $a < b$ and $b > a$ mean the same thing. It does not matter whether we write $2 < 3$ or $3 > 2$.

Furthermore, if $a < b$ or if $b > a$, then the difference $b - a$ is positive. Do you see why?

An **inequality** is a statement in which two expressions are related by an inequality symbol. The expressions are referred to as the **sides** of the inequality. Statements of the form $a < b$ or $b > a$ are called **strict inequalities,** whereas statements of the form $a \leq b$ or $b \geq a$ are called **nonstrict inequalities.**

Based on the discussion so far, we conclude that

$$a > 0 \quad \text{is equivalent to } a \text{ is positive}$$
$$a < 0 \quad \text{is equivalent to } a \text{ is negative}$$

We sometimes read $a > 0$ by saying that "a is positive." If $a \geq 0$, then either $a > 0$ or $a = 0$, and we may read this as "a is nonnegative."

Now Work PROBLEMS 25 AND 35

EXAMPLE 4

Graphing Inequalities

(a) On the real number line, graph all numbers x for which $x > 4$.
(b) On the real number line, graph all numbers x for which $x \leq 5$.

Solution

(a) See Figure 8. Notice that we use a left parenthesis to indicate that the number 4 is *not* part of the graph.
(b) See Figure 9. Notice that we use a right bracket to indicate that the number 5 *is* part of the graph.

Figure 8

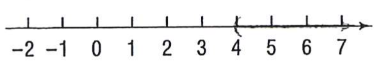
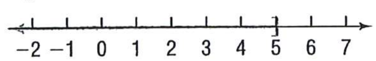


Figure 9



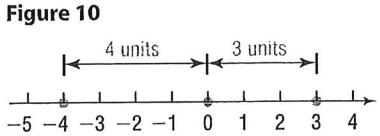
Now Work PROBLEM 41

3 Find Distance on the Real Number Line

The *absolute value* of a number a is the distance from 0 to a on the number line. For example, -4 is 4 units from 0, and 3 is 3 units from 0. See Figure 10. Thus, the absolute value of -4 is 4, and the absolute value of 3 is 3.

A more formal definition of absolute value is given next.

Figure 10



DEFINITION

The **absolute value** of a real number a, denoted by the symbol $|a|$, is defined by the rules

$$|a| = a \quad \text{if } a \geq 0 \qquad \text{and} \qquad |a| = -a \quad \text{if } a < 0$$

For example, since $-4 < 0$, the second rule must be used to get $|-4| = -(-4) = 4$.

EXAMPLE 5

Computing Absolute Value

(a) $|8| = 8$ (b) $|0| = 0$ (c) $|-15| = -(-15) = 15$

Look again at Figure 10. The distance from -4 to 3 is 7 units. This distance is the difference $3 - (-4)$, obtained by subtracting the smaller coordinate from the larger. However, since $|3 - (-4)| = |7| = 7$ and $|-4 - 3| = |-7| = 7$, we can use absolute value to calculate the distance between two points without being concerned about which is smaller.

DEFINITION

If P and Q are two points on a real number line with coordinates a and b, respectively, the **distance between P and Q,** denoted by $d(P, Q)$, is

$$d(P, Q) = |b - a|$$

Since $|b - a| = |a - b|$, it follows that $d(P, Q) = d(Q, P)$.

EXAMPLE 6

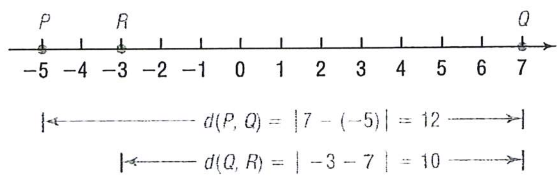
Finding Distance on a Number Line

Let P, Q, and R be points on a real number line with coordinates -5, 7, and -3, respectively. Find the distance

(a) between P and Q (b) between Q and R

Solution See Figure 11.

Figure 11



(a) $d(P, Q) = |7 - (-5)| = |12| = 12$

(b) $d(Q, R) = |-3 - 7| = |-10| = 10$

Now Work PROBLEM 47

4 Evaluate Algebraic Expressions

In algebra we use letters such as $x, y, a, b,$ and c to represent numbers. If the letter used is to represent *any* number from a given set of numbers, it is called a **variable.** A **constant** is either a fixed number, such as 5 or $\sqrt{3}$, or a letter that represents a fixed (possibly unspecified) number.

Constants and variables are combined using the operations of addition, subtraction, multiplication, and division to form *algebraic expressions*. Examples of algebraic expressions include

$$x + 3 \qquad \frac{3}{1 - t} \qquad 7x - 2y$$

To evaluate an algebraic expression, substitute for each variable its numerical value.

EXAMPLE 7

Evaluating an Algebraic Expression

Evaluate each expression if $x = 3$ and $y = -1$.

(a) $x + 3y$ (b) $5xy$ (c) $\dfrac{3y}{2 - 2x}$ (d) $|-4x + y|$

Solution (a) Substitute 3 for x and -1 for y in the expression $x + 3y$.

$$x + 3y \underset{\substack{\uparrow \\ x = 3,\, y = -1}}{=} 3 + 3(-1) = 3 + (-3) = 0$$

(b) If $x = 3$ and $y = -1$, then

$$5xy = 5(3)(-1) = -15$$

(c) If $x = 3$ and $y = -1$, then

$$\frac{3y}{2 - 2x} = \frac{3(-1)}{2 - 2(3)} = \frac{-3}{2 - 6} = \frac{-3}{-4} = \frac{3}{4}$$

(d) If $x = 3$ and $y = -1$, then

$$|-4x + y| = |-4(3) + (-1)| = |-12 + (-1)| = |-13| = 13$$

Now Work PROBLEMS 49 AND 57

5 Determine the Domain of a Variable

In working with expressions or formulas involving variables, the variables may be allowed to take on values from only a certain set of numbers. For example, in the formula for the area A of a circle of radius r, $A = \pi r^2$, the variable r is necessarily restricted to the positive real numbers. In the expression $\frac{1}{x}$, the variable x cannot take on the value 0, since division by 0 is not defined.

DEFINITION

The set of values that a variable may assume is called the **domain of the variable.**

EXAMPLE 8

Finding the Domain of a Variable

The domain of the variable x in the expression

$$\frac{5}{x - 2}$$

is $\{x | x \neq 2\}$, since, if $x = 2$, the denominator becomes 0, which is not defined.

EXAMPLE 9

Circumference of a Circle

In the formula for the circumference C of a circle of radius r,

$$C = 2\pi r$$

the domain of the variable r, representing the radius of the circle, is the set of positive real numbers. The domain of the variable C, representing the circumference of the circle, is also the set of positive real numbers.

In describing the domain of a variable, we may use either set notation or words, whichever is more convenient.

Now Work PROBLEM 67

6 Use the Laws of Exponents

Integer exponents provide a shorthand notation for representing repeated multiplications of a real number. For example,

$$2^3 = 2 \cdot 2 \cdot 2 = 8 \qquad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

DEFINITION

If a is a real number and n is a positive integer, then the symbol a^n represents the product of n factors of a. That is,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}} \qquad (1)$$

Here it is understood that $a^1 = a$.

Then $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, and so on. In the expression a^n, a is called the **base** and n is called the **exponent,** or **power.** We read a^n as "a raised to the power n" or as "a to the nth power." We usually read a^2 as "a squared" and a^3 as "a cubed."

In working with exponents, the operation of *raising to a power* is performed before any other operation. As examples,

$$4 \cdot 3^2 = 4 \cdot 9 = 36 \qquad 2^2 + 3^2 = 4 + 9 = 13$$

$$-2^4 = -16 \qquad 5 \cdot 3^2 + 2 \cdot 4 = 5 \cdot 9 + 2 \cdot 4 = 45 + 8 = 53$$

Parentheses are used to indicate operations to be performed first. For example,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16 \qquad (2 + 3)^2 = 5^2 = 25$$

DEFINITION

If $a \neq 0$, we define

$$a^0 = 1 \quad \text{if } a \neq 0$$

DEFINITION

If $a \neq 0$ and if n is a positive integer, then we define

$$a^{-n} = \frac{1}{a^n} \quad \text{if } a \neq 0$$

Whenever you encounter a negative exponent, think "reciprocal."

EXAMPLE 10

Evaluating Expressions Containing Negative Exponents

(a) $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$ (b) $x^{-4} = \dfrac{1}{x^4}$ (c) $\left(\dfrac{1}{5}\right)^{-2} = \dfrac{1}{\left(\dfrac{1}{5}\right)^2} = \dfrac{1}{\dfrac{1}{25}} = 25$

Now Work PROBLEMS 85 AND 105

The following properties, called the **Laws of Exponents,** can be proved using the preceding definitions. In the list, a and b are real numbers, and m and n are integers.

THEOREM

Laws of Exponents

$$a^m a^n = a^{m+n} \qquad (a^m)^n = a^{mn} \qquad (ab)^n = a^n b^n$$

$$\frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} \quad \text{if } a \neq 0 \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} \quad \text{if } b \neq 0$$

EXAMPLE 11 **Using the Laws of Exponents**

Write each expression so that all exponents are positive.

(a) $\dfrac{x^5y^{-2}}{x^3y}$ $x \neq 0,\ y \neq 0$ (b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2}$ $x \neq 0,\ y \neq 0$

Solution

(a) $\dfrac{x^5y^{-2}}{x^3y} = \dfrac{x^5}{x^3}\cdot\dfrac{y^{-2}}{y} = x^{5-3}\cdot y^{-2-1} = x^2y^{-3} = x^2\cdot\dfrac{1}{y^3} = \dfrac{x^2}{y^3}$

(b) $\left(\dfrac{x^{-3}}{3y^{-1}}\right)^{-2} = \dfrac{(x^{-3})^{-2}}{(3y^{-1})^{-2}} = \dfrac{x^6}{3^{-2}(y^{-1})^{-2}} = \dfrac{x^6}{\frac{1}{9}y^2} = \dfrac{9x^6}{y^2}$

Now Work PROBLEMS 87 AND 97

7 Evaluate Square Roots

In Words

$\sqrt{36}$ means "give me the nonnegative number whose square is 36."

A real number is squared when it is raised to the power 2. The inverse of squaring is finding a **square root.** For example, since $6^2 = 36$ and $(-6)^2 = 36$, the numbers 6 and -6 are square roots of 36.

The symbol $\sqrt{}$, called a **radical sign,** is used to denote the **principal,** or nonnegative, square root. For example, $\sqrt{36} = 6$.

DEFINITION If a is a nonnegative real number, the nonnegative number b, such that $b^2 = a$, is the **principal square root** of a and is denoted by $b = \sqrt{a}$.

The following comments are noteworthy:

1. Negative numbers do not have square roots (in the real number system), because the square of any real number is *nonnegative.* For example, $\sqrt{-4}$ is not a real number, because there is no real number whose square is -4.
2. The principal square root of 0 is 0, since $0^2 = 0$. That is, $\sqrt{0} = 0$.
3. The principal square root of a positive number is positive.
4. If $c \geq 0$, then $(\sqrt{c})^2 = c$. For example, $(\sqrt{2})^2 = 2$ and $(\sqrt{3})^2 = 3$.

EXAMPLE 12 **Evaluating Square Roots**

(a) $\sqrt{64} = 8$ (b) $\sqrt{\dfrac{1}{16}} = \dfrac{1}{4}$ (c) $(\sqrt{1.4})^2 = 1.4$

Examples 12(a) and (b) are examples of square roots of perfect squares, since $64 = 8^2$ and $\dfrac{1}{16} = \left(\dfrac{1}{4}\right)^2$.

Consider the expression $\sqrt{a^2}$. Since $a^2 \geq 0$, the principal square root of a^2 is defined whether $a > 0$ or $a < 0$. However, since the principal square root is nonnegative, we need an absolute value to ensure the nonnegative result. That is,

$$\sqrt{a^2} = |a| \quad a \text{ any real number} \qquad (2)$$

EXAMPLE 13 **Simplifying Expressions Using Equation 2**

(a) $\sqrt{(2.3)^2} = |2.3| = 2.3$ (b) $\sqrt{(-2.3)^2} = |-2.3| = 2.3$ (c) $\sqrt{x^2} = |x|$

Now Work PROBLEM 93

Calculators

Calculators are finite machines. As a result, they are incapable of displaying decimals that contain a large number of digits. For example, some calculators are capable of displaying only eight digits. When a number requires more than eight digits, the calculator either truncates or rounds. To see how your calculator handles decimals, divide 2 by 3. How many digits do you see? Is the last digit a 6 or a 7? If it is a 6, your calculator truncates; if it is a 7, your calculator rounds.

There are different kinds of calculators. An **arithmetic** calculator can only add, subtract, multiply, and divide numbers; therefore, this type is not adequate for this course. **Scientific** calculators have all the capabilities of arithmetic calculators and also contain **function keys** labeled ln, log, sin, cos, tan, x^y, inv, and so on. **Graphing** calculators have all the capabilities of scientific calculators and contain a screen on which graphs can be displayed. We use the symbol whenever a graphing calculator needs to be used. In this book the use of a graphing calculator is optional.

8 Use a Calculator to Evaluate Exponents

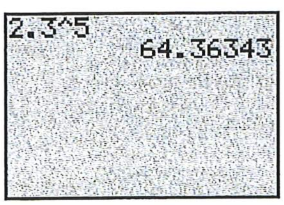
Your calculator has either a caret key, [^], or an [x^y] key, which is used for computations involving exponents.

EXAMPLE 14 **Exponents on a Graphing Calculator**

Evaluate: $(2.3)^5$

Solution Figure 12 shows the result using a TI-84 graphing calculator.

Figure 12



Now Work PROBLEM 123

A.1 Assess Your Understanding

Concepts and Vocabulary

1. A(n) ________ is a letter used in algebra to represent any number from a given set of numbers.
2. On the real number line, the real number zero is the coordinate of the ________.
3. An inequality of the form $a > b$ is called a(n) ______ inequality.
4. In the expression 2^4, the number 2 is called the ______ and 4 is called the ________________.
5. ***True or False*** The product of two negative real numbers is always greater than zero.
6. ***True or False*** The distance between two distinct points on the real number line is always greater than zero.
7. ***True or False*** The absolute value of a real number is always greater than zero.
8. ***True or False*** To multiply two expressions having the same base, retain the base and multiply the exponents.

Skill Building

In Problems 9–20, use U = universal set = $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 3, 4, 5, 9\}$, $B = \{2, 4, 6, 7, 8\}$, and $C = \{1, 3, 4, 6\}$ to find each set.

9. $A \cup B$ **10.** $A \cup C$ **11.** $A \cap B$ **12.** $A \cap C$

13. $(A \cup B) \cap C$ **14.** $(A \cap B) \cup C$ **15.** $\overline{A}$ **16.** $\overline{C}$

17. $\overline{A \cap B}$ **18.** $\overline{B \cup C}$ **19.** $\overline{A} \cup \overline{B}$ **20.** $\overline{B} \cap \overline{C}$

21. On the real number line, label the points with coordinates $0, 1, -1, \frac{5}{2}, -2.5, \frac{3}{4}$, and 0.25.

22. On the real number line, label the points with coordinates $0, -2, 2, -1.5, \frac{3}{2}, \frac{1}{3}$, and $\frac{2}{3}$.

In Problems 23–32, replace the question mark by $<$, $>$, or $=$, whichever is correct.

23. $\frac{1}{2}$? 0 **24.** 5 ? 6 **25.** -1 ? -2 **26.** -3 ? $-\frac{5}{2}$ **27.** π ? 3.14

28. $\sqrt{2}$? 1.41 **29.** $\frac{1}{2}$? 0.5 **30.** $\frac{1}{3}$? 0.33 **31.** $\frac{2}{3}$? 0.67 **32.** $\frac{1}{4}$? 0.25

In Problems 33–38, write each statement as an inequality.

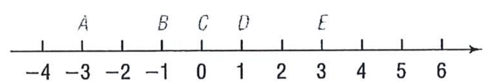
33. x is positive **34.** z is negative **35.** x is less than 2

36. y is greater than -5 **37.** x is less than or equal to 1 **38.** x is greater than or equal to 2

In Problems 39–42, graph the numbers x on the real number line.

39. $x \geq -2$ **40.** $x < 4$ **41.** $x > -1$ **42.** $x \leq 7$

In Problems 43–48, use the given real number line to compute each distance.



43. $d(C, D)$ **44.** $d(C, A)$ **45.** $d(D, E)$ **46.** $d(C, E)$ **47.** $d(A, E)$ **48.** $d(D, B)$

In Problems 49–56, evaluate each expression if $x = -2$ and $y = 3$.

49. $x + 2y$ **50.** $3x + y$ **51.** $5xy + 2$ **52.** $-2x + xy$

53. $\dfrac{2x}{x - y}$ **54.** $\dfrac{x + y}{x - y}$ **55.** $\dfrac{3x + 2y}{2 + y}$ **56.** $\dfrac{2x - 3}{y}$

In Problems 57–66, find the value of each expression if $x = 3$ and $y = -2$.

57. $|x + y|$ **58.** $|x - y|$ **59.** $|x| + |y|$ **60.** $|x| - |y|$ **61.** $\dfrac{|x|}{x}$

62. $\dfrac{|y|}{y}$ **63.** $|4x - 5y|$ **64.** $|3x + 2y|$ **65.** $||4x| - |5y||$ **66.** $3|x| + 2|y|$

In Problems 67–74, determine which of the value(s) (a) through (d), if any, must be excluded from the domain of the variable in each expression:

(a) $x = 3$ *(b)* $x = 1$ *(c)* $x = 0$ *(d)* $x = -1$

67. $\dfrac{x^2 - 1}{x}$ **68.** $\dfrac{x^2 + 1}{x}$ **69.** $\dfrac{x}{x^2 - 9}$ **70.** $\dfrac{x}{x^2 + 9}$

71. $\dfrac{x^2}{x^2 + 1}$ **72.** $\dfrac{x^3}{x^2 - 1}$ **73.** $\dfrac{x^2 + 5x - 10}{x^3 - x}$ **74.** $\dfrac{-9x^2 - x + 1}{x^3 + x}$

In Problems 75–78, determine the domain of the variable x in each expression.

75. $\dfrac{4}{x-5}$ **76.** $\dfrac{-6}{x+4}$ **77.** $\dfrac{x}{x+4}$ **78.** $\dfrac{x-2}{x-6}$

In Problems 79–82, use the formula $C = \dfrac{5}{9}(F - 32)$ *for converting degrees Fahrenheit into degrees Celsius to find the Celsius measure of each Fahrenheit temperature.*

79. $F = 32°$ **80.** $F = 212°$ **81.** $F = 77°$ **82.** $F = -4°$

In Problems 83–94, simplify each expression.

83. $(-4)^2$ **84.** -4^2 **85.** 4^{-2} **86.** -4^{-2} **87.** $3^{-6} \cdot 3^4$ **88.** $4^{-2} \cdot 4^3$

89. $(3^{-2})^{-1}$ **90.** $(2^{-1})^{-3}$ **91.** $\sqrt{25}$ **92.** $\sqrt{36}$ **93.** $\sqrt{(-4)^2}$ **94.** $\sqrt{(-3)^2}$

In Problems 95–104, simplify each expression. Express the answer so that all exponents are positive. Whenever an exponent is 0 or negative, we assume that the base is not 0.

95. $(8x^3)^2$ **96.** $(-4x^2)^{-1}$ **97.** $(x^2y^{-1})^2$ **98.** $(x^{-1}y)^3$ **99.** $\dfrac{x^2y^3}{xy^4}$

100. $\dfrac{x^{-2}y}{xy^2}$ **101.** $\dfrac{(-2)^3x^4(yz)^2}{3^2xy^3z}$ **102.** $\dfrac{4x^{-2}(yz)^{-1}}{2^3x^4y}$ **103.** $\left(\dfrac{3x^{-1}}{4y^{-1}}\right)^{-2}$ **104.** $\left(\dfrac{5x^{-2}}{6y^{-2}}\right)^{-3}$

In Problems 105–116, find the value of each expression if $x = 2$ *and* $y = -1$.

105. $2xy^{-1}$ **106.** $-3x^{-1}y$ **107.** $x^2 + y^2$ **108.** x^2y^2

109. $(xy)^2$ **110.** $(x + y)^2$ **111.** $\sqrt{x^2}$ **112.** $(\sqrt{x})^2$

113. $\sqrt{x^2 + y^2}$ **114.** $\sqrt{x^2} + \sqrt{y^2}$ **115.** x^y **116.** y^x

117. Find the value of the expression $2x^3 - 3x^2 + 5x - 4$ if $x = 2$. What is the value if $x = 1$?

118. Find the value of the expression $4x^3 + 3x^2 - x + 2$ if $x = 1$. What is the value if $x = 2$?

119. What is the value of $\dfrac{(666)^4}{(222)^4}$?

120. What is the value of $(0.1)^3(20)^3$?

In Problems 121–128, use a calculator to evaluate each expression. Round your answer to three decimal places.

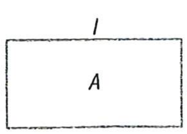
121. $(8.2)^6$ **122.** $(3.7)^5$ **123.** $(6.1)^{-3}$ **124.** $(2.2)^{-5}$

125. $(-2.8)^6$ **126.** $-(2.8)^6$ **127.** $(-8.11)^{-4}$ **128.** $-(8.11)^{-4}$

Applications and Extensions

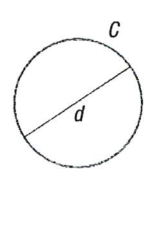
In Problems 129–138, express each statement as an equation involving the indicated variables.

129. Area of a Rectangle The area A of a rectangle is the product of its length l and its width w.

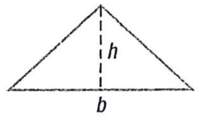


130. Perimeter of a Rectangle The perimeter P of a rectangle is twice the sum of its length l and its width w.

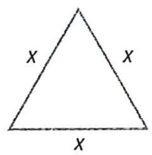
131. Circumference of a Circle The circumference C of a circle is the product of π and its diameter d.



132. Area of a Triangle The area A of a triangle is one-half the product of its base b and its height h.

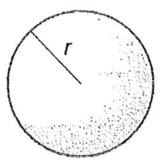


133. Area of an Equilateral Triangle The area A of an equilateral triangle is $\frac{\sqrt{3}}{4}$ times the square of the length x of one side.



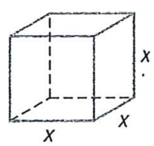
134. Perimeter of an Equilateral Triangle The perimeter P of an equilateral triangle is 3 times the length x of one side.

135. Volume of a Sphere The volume V of a sphere is $\frac{4}{3}$ times π times the cube of the radius r.



136. Surface Area of a Sphere The surface area S of a sphere is 4 times π times the square of the radius r.

137. Volume of a Cube The volume V of a cube is the cube of the length x of a side.



138. Surface Area of a Cube The surface area S of a cube is 6 times the square of the length x of a side.

139. Manufacturing Cost The weekly production cost C of manufacturing x watches is given by the formula $C = 4000 + 2x$, where the variable C is in dollars.

(a) What is the cost of producing 1000 watches?

(b) What is the cost of producing 2000 watches?

140. Balancing a Checkbook At the beginning of the month, Mike had a balance of \$210 in his checking account. During the next month, he deposited \$80, wrote a check for \$120, made another deposit of \$25, and wrote two checks: one for \$60 and the other for \$32. He was also assessed a monthly service charge of \$5. What was his balance at the end of the month?

In Problems 141 and 142, write an inequality using an absolute value to describe each statement.

141. x is at least 6 units from 4.

142. x is more than 5 units from 2.

143. U.S. Voltage In the United States, normal household voltage is 110 volts. It is acceptable for the actual voltage x to differ from normal by at most 5 volts. A formula that describes this is

$$|x - 110| \le 5$$

(a) Show that a voltage of 108 volts is acceptable.

(b) Show that a voltage of 104 volts is not acceptable.

144. Foreign Voltage In some countries, normal household voltage is 220 volts. It is acceptable for the actual voltage x to differ from normal by at most 8 volts. A formula that describes this is

$$|x - 220| \le 8$$

(a) Show that a voltage of 214 volts is acceptable.

(b) Show that a voltage of 209 volts is not acceptable.

145. Making Precision Ball Bearings The FireBall Company manufactures ball bearings for precision equipment. One of its products is a ball bearing with a stated radius of 3 centimeters (cm). Only ball bearings with a radius within 0.01 cm of this stated radius are acceptable. If x is the radius of a ball bearing, a formula describing this situation is

$$|x - 3| \le 0.01$$

(a) Is a ball bearing of radius $x = 2.999$ acceptable?

(b) Is a ball bearing of radius $x = 2.89$ acceptable?

146. Body Temperature Normal human body temperature is 98.6°F. A temperature x that differs from normal by at least 1.5°F is considered unhealthy. A formula that describes this is

$$|x - 98.6| \ge 1.5$$

(a) Show that a temperature of 97°F is unhealthy.

(b) Show that a temperature of 100°F is not unhealthy.

147. Does $\frac{1}{3}$ equal 0.333? If not, which is larger? By how much?

148. Does $\frac{2}{3}$ equal 0.666? If not, which is larger? By how much?

Explaining Concepts: Discussion and Writing

149. Is there a positive real number "closest" to 0?

150. **Number Game** I'm thinking of a number! It lies between 1 and 10; its square is rational and lies between 1 and 10. The number is larger than π. Correct to two decimal places (that is, truncated to two decimal places) name the number. Now think of your own number, describe it, and challenge a fellow student to name it.

151. Write a brief paragraph that illustrates the similarities and differences between "less than" ($<$) and "less than or equal to" ($\le$).

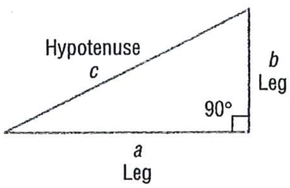
152. Give a reason why the statement $5 < 8$ is true.

A.2 Geometry Essentials

OBJECTIVES
1 Use the Pythagorean Theorem and Its Converse (p. A14)
2 Know Geometry Formulas (p. A15)
3 Understand Congruent Triangles and Similar Triangles (p. A16)

1 Use the Pythagorean Theorem and Its Converse

Figure 13



The *Pythagorean Theorem* is a statement about *right triangles*. A **right triangle** is one that contains a **right angle,** that is, an angle of 90°. The side of the triangle opposite the 90° angle is called the **hypotenuse;** the remaining two sides are called **legs.** In Figure 13 we have used c to represent the length of the hypotenuse and a and b to represent the lengths of the legs. Notice the use of the symbol ⌐ to show the 90° angle. We now state the Pythagorean Theorem.

PYTHAGOREAN THEOREM

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. That is, in the right triangle shown in Figure 13,

$$c^2 = a^2 + b^2 \quad \textbf{(1)}$$

EXAMPLE 1 **Finding the Hypotenuse of a Right Triangle**

In a right triangle, one leg has length 4 and the other has length 3. What is the length of the hypotenuse?

Solution Since the triangle is a right triangle, we use the Pythagorean Theorem with $a = 4$ and $b = 3$ to find the length c of the hypotenuse. From equation (1), we have

$$c^2 = a^2 + b^2$$
$$c^2 = 4^2 + 3^2 = 16 + 9 = 25$$
$$c = \sqrt{25} = 5$$

Now Work PROBLEM 13

The converse of the Pythagorean Theorem is also true.

CONVERSE OF THE PYTHAGOREAN THEOREM

In a triangle, if the square of the length of one side equals the sum of the squares of the lengths of the other two sides, the triangle is a right triangle. The 90° angle is opposite the longest side.

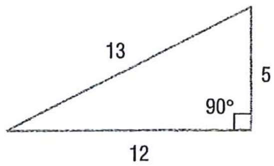
EXAMPLE 2 **Verifying That a Triangle Is a Right Triangle**

Show that a triangle whose sides are of lengths 5, 12, and 13 is a right triangle. Identify the hypotenuse.

Solution We square the lengths of the sides.

$$5^2 = 25, \qquad 12^2 = 144, \qquad 13^2 = 169$$

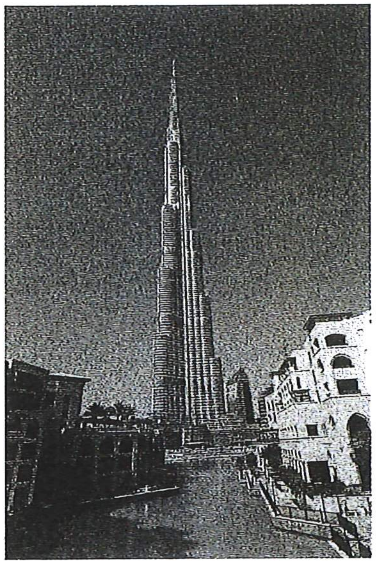
Figure 14



Notice that the sum of the first two squares (25 and 144) equals the third square (169). Hence, the triangle is a right triangle. The longest side, 13, is the hypotenuse. See Figure 14.

Now Work PROBLEM 21

EXAMPLE 3 **Applying the Pythagorean Theorem**



The tallest building in the world is Burj Khalifa in Dubai, United Arab Emirates, at 2717 feet and 160 floors. The observation deck is 1450 feet above ground level. How far can a person standing on the observation deck see (with the aid of a telescope)? Use 3960 miles for the radius of Earth.

Source: Wikipedia 2010

Solution From the center of Earth, draw two radii: one through Burj Khalifa and the other to the farthest point a person can see from the observation deck. See Figure 15. Apply the Pythagorean Theorem to the right triangle.

Since 1 mile = 5280 feet, then 1450 feet $= \dfrac{1450}{5280}$ mile. So we have

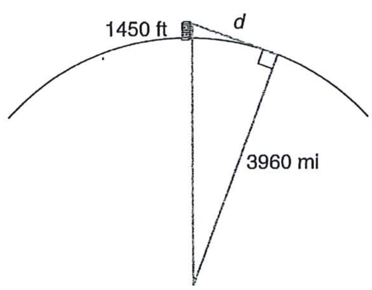
$$d^2 + (3960)^2 = \left(3960 + \frac{1450}{5280}\right)^2$$

$$d^2 = \left(3960 + \frac{1450}{5280}\right)^2 - (3960)^2 \approx 2175.08$$

$$d \approx 46.64$$

A person can see almost 47 miles from the observation tower.

Figure 15

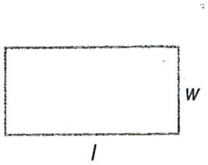


Now Work PROBLEM 53

2 Know Geometry Formulas

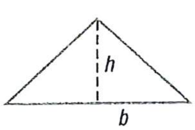
Certain formulas from geometry are useful in solving algebra problems.

For a rectangle of length l and width w,

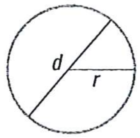


$$\text{Area} = lw \qquad \text{Perimeter} = 2l + 2w$$

For a triangle with base b and altitude h,

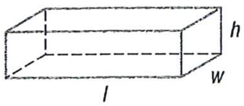


$$\text{Area} = \frac{1}{2}bh$$



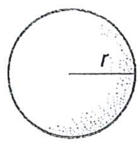
For a circle of radius r (diameter $d = 2r$),

Area $= \pi r^2$	Circumference $= 2\pi r = \pi d$



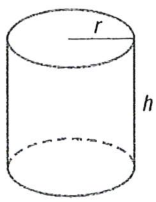
For a closed rectangular box of length l, width w, and height h,

Volume $= lwh$	Surface area $= 2lh + 2wh + 2lw$



For a sphere of radius r,

Volume $= \frac{4}{3}\pi r^3$	Surface area $= 4\pi r^2$



For a right circular cylinder of height h and radius r,

Volume $= \pi r^2 h$	Surface area $= 2\pi r^2 + 2\pi rh$

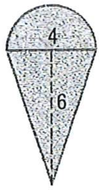
Now Work PROBLEM 29

EXAMPLE 4 **Using Geometry Formulas**

A Christmas tree ornament is in the shape of a semicircle on top of a triangle. How many square centimeters (cm) of copper is required to make the ornament if the height of the triangle is 6 cm and the base is 4 cm?

Solution See Figure 16. The amount of copper required equals the shaded area. This area is the sum of the areas of the triangle and the semicircle. The triangle has height $h = 6$ and base $b = 4$. The semicircle has diameter $d = 4$, so its radius is $r = 2$.

Figure 16



$$\begin{aligned} \text{Area} &= \text{Area of triangle} + \text{Area of semicircle} \\ &= \frac{1}{2}bh + \frac{1}{2}\pi r^2 = \frac{1}{2}(4)(6) + \frac{1}{2}\pi \cdot 2^2 \qquad b = 4;\ h = 6;\ r = 2 \\ &= 12 + 2\pi \approx 18.28\ \text{cm}^2 \end{aligned}$$

About 18.28 cm^2 of copper is required.

Now Work PROBLEM 47

3 Understand Congruent Triangles and Similar Triangles

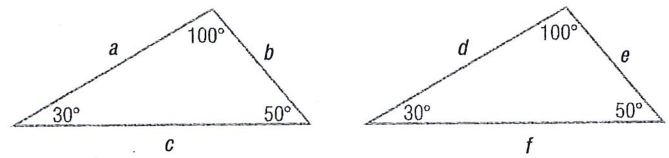
In Words
Two triangles are congruent if they have the same size and shape.

Throughout the text we will make reference to triangles. We begin with a discussion of *congruent* triangles. According to dictionary.com, the word **congruent** means coinciding exactly when superimposed. For example, two angles are congruent if they have the same measure and two line segments are congruent if they have the same length.

DEFINITION Two triangles are **congruent** if each of the corresponding angles is the same measure and each of the corresponding sides is the same length.

In Figure 17, corresponding angles are equal and the lengths of the corresponding sides are equal: $a = d$, $b = e$, and $c = f$. We conclude that these triangles are congruent.

Figure 17 Congruent triangles

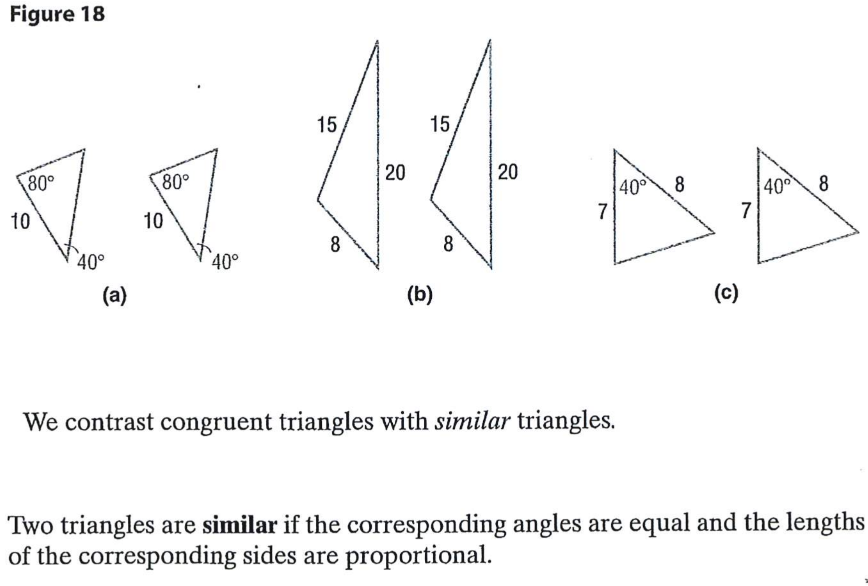


It is not necessary to verify that all three angles and all three sides are the same measure to determine whether two triangles are congruent.

Determining Congruent Triangles

1. **Angle–Side–Angle Case** Two triangles are congruent if two of the angles are equal and the lengths of the corresponding sides between the two angles are equal.
   For example, in Figure 18(a), the two triangles are congruent because two angles and the included side are equal.
2. **Side–Side–Side Case** Two triangles are congruent if the lengths of the corresponding sides of the triangles are equal.
   For example, in Figure 18(b), the two triangles are congruent because the three corresponding sides are all equal.
3. **Side–Angle–Side Case** Two triangles are congruent if the lengths of two corresponding sides are equal and the angles between the two sides are the same.
   For example, in Figure 18(c), the two triangles are congruent because two sides and the included angle are equal.

Figure 18



We contrast congruent triangles with *similar* triangles.

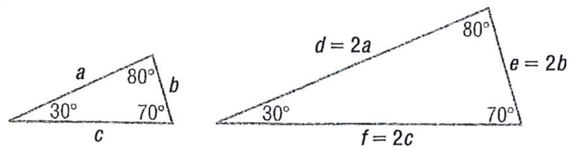
DEFINITION

Two triangles are **similar** if the corresponding angles are equal and the lengths of the corresponding sides are proportional.

In Words
Two triangles are similar if they have the same shape, but (possibly) different sizes.

For example, the triangles in Figure 19 are similar because the corresponding angles are equal. In addition, the lengths of the corresponding sides are proportional because each side in the triangle on the right is twice as long as each corresponding side in the triangle on the left. That is, the ratio of the corresponding sides is a constant: $\frac{d}{a} = \frac{e}{b} = \frac{f}{c} = 2$.

Figure 19

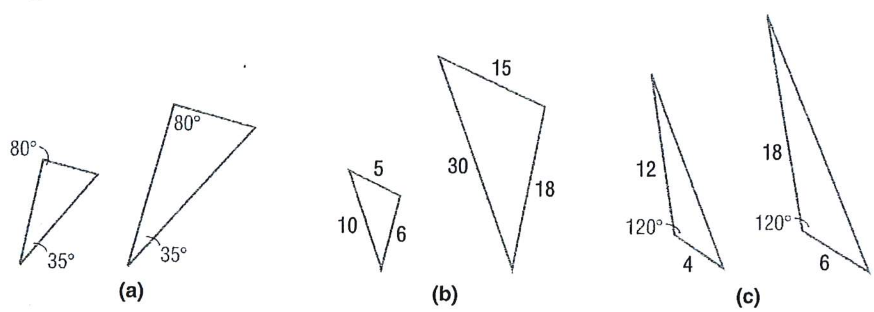


It is not necessary to verify that all three angles are equal and all three sides are proportional to determine whether two triangles are congruent.

Determining Similar Triangles

1. **Angle–Angle Case** Two triangles are similar if two of the corresponding angles are equal.
   For example, in Figure 20(a), the two triangles are similar because two angles are equal.
2. **Side–Side–Side Case** Two triangles are similar if the lengths of all three sides of each triangle are proportional.
   For example, in Figure 20(b), the two triangles are similar because
   $$\frac{10}{30} = \frac{5}{15} = \frac{6}{18} = \frac{1}{3}.$$
3. **Side–Angle–Side Case** Two triangles are similar if two corresponding sides are proportional and the angles between the two sides are equal.
   For example, in Figure 20(c), the two triangles are similar because $\frac{4}{6} = \frac{12}{18} = \frac{2}{3}$ and the angles between the sides are equal.

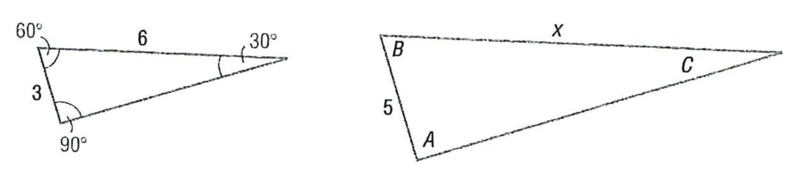
Figure 20



EXAMPLE 5 **Using Similar Triangles**

Given that the triangles in Figure 21 are similar, find the missing length x and the angles A, B, and C.

Figure 21



Solution Because the triangles are similar, corresponding angles are equal. So $A = 90°$, $B = 60°$, and $C = 30°$. Also, the corresponding sides are proportional. That is, $\frac{3}{5} = \frac{6}{x}$. We solve this equation for x.

$$\frac{3}{5} = \frac{6}{x}$$

$$5x \cdot \frac{3}{5} = 5x \cdot \frac{6}{x} \qquad \text{Multiply both sides by } 5x.$$

$$3x = 30 \qquad \text{Simplify.}$$

$$x = 10 \qquad \text{Divide both sides by 3.}$$

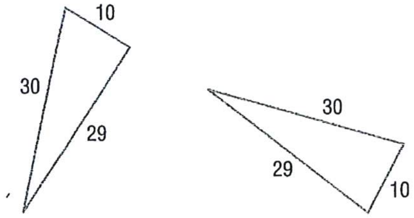
The missing length is 10 units.

Now Work PROBLEM 41

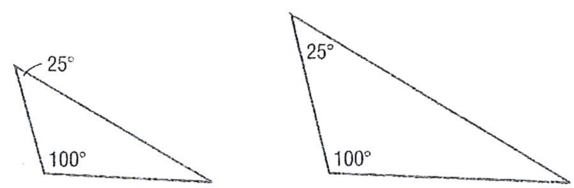
A.2 Assess Your Understanding

Concepts and Vocabulary

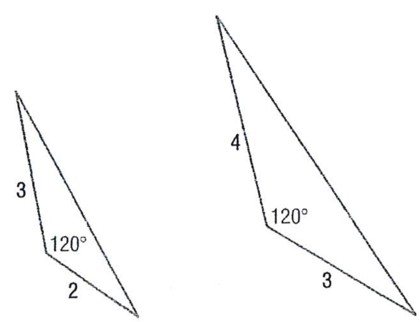
1. A(n) _______ triangle is one that contains an angle of 90 degrees. The longest side is called the ___________.
2. For a triangle with base b and altitude h, a formula for the area A is ___________.
3. The formula for the circumference C of a circle of radius r is __________.
4. Two triangles are ________ if corresponding angles are equal and the lengths of the corresponding sides are proportional.
5. ***True or False*** In a right triangle, the square of the length of the longest side equals the sum of the squares of the lengths of the other two sides.
6. ***True or False*** The triangle with sides of length 6, 8, and 10 is a right triangle.
7. ***True or False*** The volume of a sphere of radius r is $\frac{4}{3}\pi r^2$.
8. ***True or False*** The triangles shown are congruent.



9. ***True or False*** The triangles shown are similar.



10. ***True or False*** The triangles shown are similar.



Skill Building

In Problems 11–16, the lengths of the legs of a right triangle are given. Find the hypotenuse.

11. $a = 5, \quad b = 12$ **12.** $a = 6, \quad b = 8$ **13.** $a = 10, \quad b = 24$

14. $a = 4, \quad b = 3$ **15.** $a = 7, \quad b = 24$ **16.** $a = 14, \quad b = 48$

In Problems 17–24, the lengths of the sides of a triangle are given. Determine which are right triangles. For those that are, identify the hypotenuse.

17. 3, 4, 5 **18.** 6, 8, 10 **19.** 4, 5, 6 **20.** 2, 2, 3

21. 7, 24, 25 **22.** 10, 24, 26 **23.** 6, 4, 3 **24.** 5, 4, 7

25. Find the area A of a rectangle with length 4 inches and width 2 inches.

26. Find the area A of a rectangle with length 9 centimeters and width 4 centimeters.

27. Find the area A of a triangle with height 4 inches and base 2 inches.

28. Find the area A of a triangle with height 9 centimeters and base 4 centimeters.

29. Find the area A and circumference C of a circle of radius 5 meters.

30. Find the area A and circumference C of a circle of radius 2 feet.

31. Find the volume V and surface area S of a rectangular box with length 8 feet, width 4 feet, and height 7 feet.

32. Find the volume V and surface area S of a rectangular box with length 9 inches, width 4 inches, and height 8 inches.

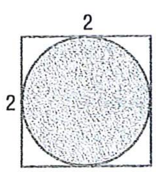
33. Find the volume V and surface area S of a sphere of radius 4 centimeters.

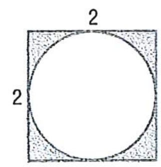
34. Find the volume V and surface area S of a sphere of radius 3 feet.

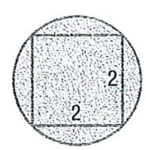
35. Find the volume V and surface area S of a right circular cylinder with radius 9 inches and height 8 inches.

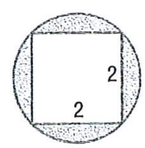
36. Find the volume V and surface area S of a right circular cylinder with radius 8 inches and height 9 inches.

In Problems 37–40, find the area of the shaded region.

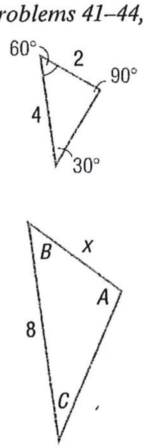
37.


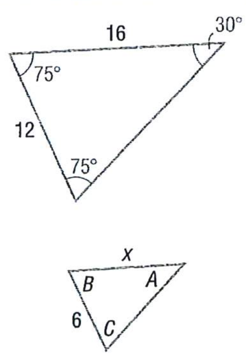
38.


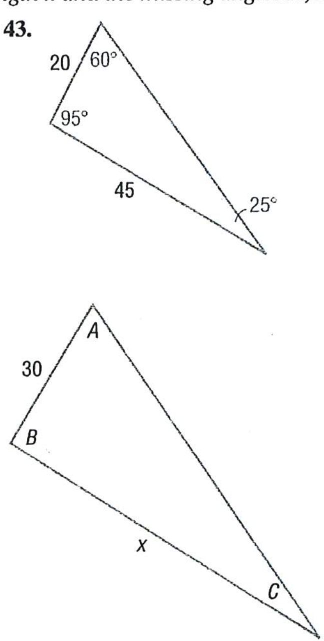
39.


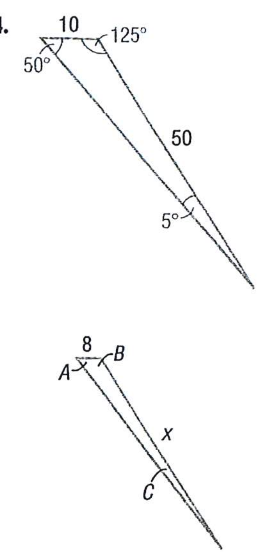
40.


In Problems 41–44, each pair of triangles is similar. Find the missing length x and the missing angles A, B, and C.

41.


42.


43.


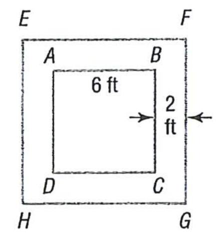
44.


Applications and Extensions

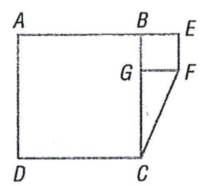
45. How many feet does a wheel with a diameter of 16 inches travel after four revolutions?

46. How many revolutions will a circular disk with a diameter of 4 feet have completed after it has rolled 20 feet?

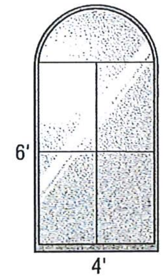
47. In the figure shown, $ABCD$ is a square, with each side of length 6 feet. The width of the border (shaded portion) between the outer square $EFGH$ and $ABCD$ is 2 feet. Find the area of the border.



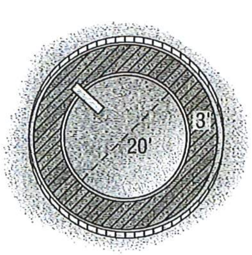
48. Refer to the figure. Square $ABCD$ has an area of 100 square feet; square $BEFG$ has an area of 16 square feet. What is the area of the triangle CGF?



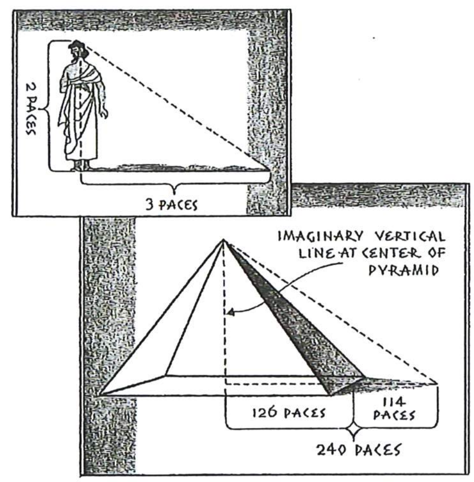
49. Architecture A **Norman window** consists of a rectangle surmounted by a semicircle. Find the area of the Norman window shown in the illustration. How much wood frame is needed to enclose the window?



50. Construction A circular swimming pool, 20 feet in diameter, is enclosed by a wooden deck that is 3 feet wide. What is the area of the deck? How much fence is required to enclose the deck?

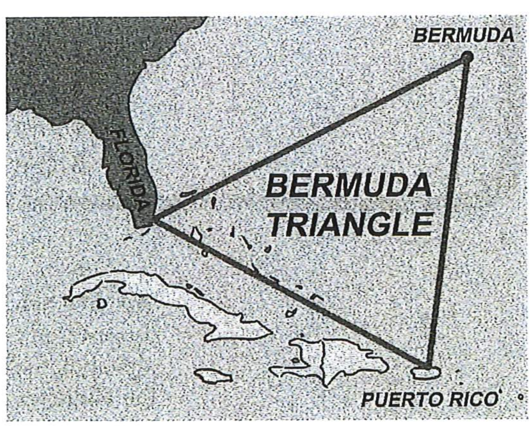


51. How Tall Is the Great Pyramid? The ancient Greek philosopher Thales of Miletus is reported on one occasion to have visited Egypt and calculated the height of the Great Pyramid of Cheops by means of shadow reckoning. Thales knew that each side of the base of the pyramid was 252 paces and that his own height was 2 paces. He measured the length of the pyramid's shadow to be 114 paces and determined the length of his shadow to be 3 paces. See the illustration. Using similar triangles, determine the height of the Great Pyramid in terms of the number of paces.



Source: Diggins, Julia E., illustrations by Corydon Bell, *String, Straightedge and Shadow: The Story of Geometry*, 2003 Whole Spirit Press, http://wholespiritpress.com.

52. The Bermuda Triangle Karen is doing research on the Bermuda Triangle, which she defines roughly by Hamilton, Bermuda; San Juan, Puerto Rico; and Fort Lauderdale, Florida. On her atlas Karen measures the straight-line distances from Hamilton to Fort Lauderdale, Fort Lauderdale to San Juan, and San Juan to Hamilton to be approximately 57 millimeters (mm), 58 mm, and 53.5 mm, respectively. If the actual distance from Fort Lauderdale to San Juan is 1046 miles, approximate the actual distances from San Juan to Hamilton and from Hamilton to Fort Lauderdale.



Source: Reprinted with permission from Red River Press, Inc., Winnipeg, Canada.

In Problems 53–55, use the facts that the radius of Earth is 3960 miles and 1 mile = 5280 feet.

53. **How Far Can You See?** The conning tower of the U.S.S. *Silversides,* a World War II submarine now permanently stationed in Muskegon, Michigan, is approximately 20 feet above sea level. How far can you see from the conning tower?

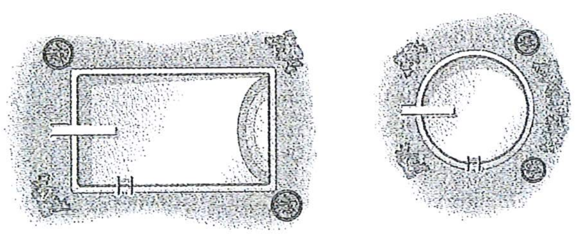
54. **How Far Can You See?** A person who is 6 feet tall is standing on the beach in Fort Lauderdale, Florida, and looks out onto the Atlantic Ocean. Suddenly, a ship appears on the horizon. How far is the ship from shore?

55. **How Far Can You See?** The deck of a destroyer is 100 feet above sea level. How far can a person see from the deck? How far can a person see from the bridge, which is 150 feet above sea level?

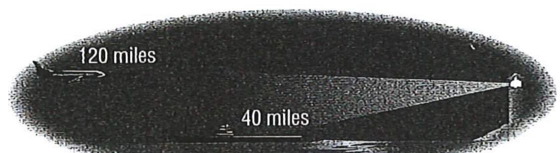
56. Suppose that m and n are positive integers with $m > n$. If $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, show that a, b, and c are the lengths of the sides of a right triangle. (This formula can be used to find the sides of a right triangle that are integers, such as 3, 4, 5; 5, 12, 13; and so on. Such triplets of integers are called **Pythagorean triples.**)

Explaining Concepts: Discussion and Writing

57. You have 1000 feet of flexible pool siding and wish to construct a swimming pool. Experiment with rectangular-shaped pools with perimeters of 1000 feet. How do their areas vary? What is the shape of the rectangle with the largest area? Now compute the area enclosed by a circular pool with a perimeter (circumference) of 1000 feet. What would be your choice of shape for the pool? If rectangular, what is your preference for dimensions? Justify your choice. If your only consideration is to have a pool that encloses the most area, what shape should you use?



58. **The Gibb's Hill Lighthouse, Southampton, Bermuda,** in operation since 1846, stands 117 feet high on a hill 245 feet high, so its beam of light is 362 feet above sea level. A brochure states that the light itself can be seen on the horizon about 26 miles from the lighthouse. Verify the correctness of this information. The brochure further states that ships 40 miles away can see the light and planes flying at 10,000 feet can see it 120 miles away. Verify the accuracy of these statements. What assumption did the brochure make about the height of the ship?



A.3 Polynomials

OBJECTIVES
1 Recognize Monomials (p. A23)
2 Recognize Polynomials (p. A23)
3 Know Formulas for Special Products (p. A24)
4 Divide Polynomials Using Long Division (p. A25)
5 Factor Polynomials (p. A28)
6 Complete the Square (p. A29)

We have described algebra as a generalization of arithmetic in which letters are used to represent real numbers. From now on, we shall use the letters at the end of the alphabet, such as x, y, and z, to represent variables and the letters at the beginning of the alphabet, such as a, b, and c, to represent constants. In the expressions $3x + 5$ and $ax + b$, it is understood that x is a variable and that a and b are constants, even though the constants a and b are unspecified. As you will find out, the context usually makes the intended meaning clear.

1 Recognize Monomials

DEFINITION

A **monomial** in one variable is the product of a constant and a variable raised to a nonnegative integer power. A monomial is of the form

COMMENT The nonnegative integers are the integers 0, 1, 2, 3, ...

$$ax^k$$

where a is a constant, x is a variable, and $k \geq 0$ is an integer. The constant a is called the **coefficient** of the monomial. If $a \neq 0$, then k is the **degree** of the monomial.

EXAMPLE 1 Examples of Monomials

Monomial	Coefficient	Degree	
(a) $6x^2$	6	2	
(b) $-\sqrt{2}x^3$	$-\sqrt{2}$	3	
(c) 3	3	0	Since $3 = 3 \cdot 1 = 3x^0, x \neq 0$
(d) $-5x$	-5	1	Since $-5x = -5x^1$
(e) x^4	1	4	Since $x^4 = 1 \cdot x^4$

Now let's look at some expressions that are not monomials.

EXAMPLE 2 Examples of Nonmonomial Expressions

(a) $3x^{1/2}$ is not a monomial, since the exponent of the variable x is $\frac{1}{2}$ and $\frac{1}{2}$ is not a nonnegative integer.

(b) $4x^{-3}$ is not a monomial, since the exponent of the variable x is -3 and -3 is not a nonnegative integer.

Now Work PROBLEM 11

2 Recognize Polynomials

Two monomials with the same variable raised to the same power are called **like terms.** For example, $2x^4$ and $-5x^4$ are like terms. In contrast, the monomials $2x^3$ and $2x^5$ are not like terms.

We can add or subtract like terms using the Distributive Property. For example,

$$2x^2 + 5x^2 = (2 + 5)x^2 = 7x^2 \quad \text{and} \quad 8x^3 - 5x^3 = (8 - 5)x^3 = 3x^3$$

The sum or difference of two monomials having different degrees is called a **binomial.** The sum or difference of three monomials with three different degrees is called a **trinomial.** For example,

$x^2 - 2$ is a binomial.
$x^3 - 3x + 5$ is a trinomial.
$2x^2 + 5x^2 + 2 = 7x^2 + 2$ is a binomial.

DEFINITION

A **polynomial** in one variable is an algebraic expression of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \qquad (1)$$

In Words
A polynomial is a sum of monomials.

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are constants,* called the **coefficients** of the polynomial, $n \geq 0$ is an integer, and x is a variable. If $a_n \neq 0$, it is the **leading coefficient,** and n is the **degree** of the polynomial.

The monomials that make up a polynomial are called its **terms.** If all the coefficients are 0, the polynomial is called the **zero polynomial,** which has no degree.

Polynomials are usually written in **standard form,** beginning with the nonzero term of highest degree and continuing with terms in descending order according to degree. If a power of x is missing, it is because its coefficient is zero.

EXAMPLE 3

Examples of Polynomials

Polynomial	Coefficients	Degree
$-8x^3 + 4x^2 + 6x + 2$	$-8, 4, 6, 2$	3
$3x^2 - 5 = 3x^2 + 0 \cdot x + (-5)$	$3, 0, -5$	2
$8 - 2x + x^2 = 1 \cdot x^2 + (-2)x + 8$	$1, -2, 8$	2
$5x + \sqrt{2} = 5x^1 + \sqrt{2}$	$5, \sqrt{2}$	1
$3 = 3 \cdot 1 = 3 \cdot x^0$	3	0
0	0	No degree

Although we have been using x to represent the variable, letters such as y or z are also commonly used.

$3x^4 - x^2 + 2$ is a polynomial (in x) of degree 4.
$9y^3 - 2y^2 + y - 3$ is a polynomial (in y) of degree 3.
$z^5 + \pi$ is a polynomial (in z) of degree 5.

Algebraic expressions such as

$$\frac{1}{x} \quad \text{and} \quad \frac{x^2 + 1}{x + 5}$$

are not polynomials. The first is not a polynomial because $\frac{1}{x} = x^{-1}$ has an exponent that is not a nonnegative integer. Although the second expression is the quotient of two polynomials, the polynomial in the denominator has degree greater than 0, so the expression cannot be a polynomial.

Now Work PROBLEM 21

3 Know Formulas for Special Products

Certain products, which we call **special products,** occur frequently in algebra. For example, we can find the product of two binomials using the **FOIL** (*F*irst, *O*uter, *I*nner, *L*ast) method.

* The notation a_n is read as "a sub n." The number n is called a **subscript** and should not be confused with an exponent. We use subscripts to distinguish one constant from another when a large or undetermined number of constants is required.

$$(ax + b)(cx + d) = ax(cx + d) + b(cx + d)$$

$$= \underbrace{ax \cdot cx}_{\text{First}} + \underbrace{ax \cdot d}_{\text{Outer}} + \underbrace{b \cdot cx}_{\text{Inner}} + \underbrace{b \cdot d}_{\text{Last}}$$

$$= acx^2 + adx + bcx + bd$$

$$= acx^2 + (ad + bc)x + bd$$

EXAMPLE 4 **Using FOIL**

(a) $(x - 3)(x + 3) = \underset{F}{x^2} + \underset{O}{3x} - \underset{I}{3x} - \underset{L}{9} = x^2 - 9$

(b) $(x + 2)^2 = (x + 2)(x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4$

(c) $(x - 3)^2 = (x - 3)(x - 3) = x^2 - 3x - 3x + 9 = x^2 - 6x + 9$

(d) $(x + 3)(x + 1) = x^2 + x + 3x + 3 = x^2 + 4x + 3$

(e) $(2x + 1)(3x + 4) = 6x^2 + 8x + 3x + 4 = 6x^2 + 11x + 4$

Now Work PROBLEM 41

Some products have been given special names because of their form. In the list that follows, x, a, and b are real numbers.

Difference of Two Squares

$$(x - a)(x + a) = x^2 - a^2 \qquad \textbf{(2)}$$

Squares of Binomials, or Perfect Squares

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \textbf{(3a)}$$

$$(x - a)^2 = x^2 - 2ax + a^2 \qquad \textbf{(3b)}$$

Cubes of Binomials, or Perfect Cubes

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3 \qquad \textbf{(4a)}$$

$$(x - a)^3 = x^3 - 3ax^2 + 3a^2x - a^3 \qquad \textbf{(4b)}$$

Difference of Two Cubes

$$(x - a)(x^2 + ax + a^2) = x^3 - a^3 \qquad \textbf{(5)}$$

Sum of Two Cubes

$$(x + a)(x^2 - ax + a^2) = x^3 + a^3 \qquad \textbf{(6)}$$

Now Work PROBLEMS 45, 49, AND 53

4 Divide Polynomials Using Long Division

The procedure for dividing two polynomials is similar to the procedure for dividing two integers.

EXAMPLE 5 **Dividing Two Integers**

Divide 842 by 15.

Solution

$$\begin{array}{r l}
56 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow 15\overline{)842} & \leftarrow \text{Dividend} \\
\underline{75} & \leftarrow 5 \cdot 15 \text{ (subtract)} \\
92 & \\
\underline{90} & \leftarrow 6 \cdot 15 \text{ (subtract)} \\
2 & \leftarrow \text{Remainder}
\end{array}$$

So, $\frac{842}{15} = 56 + \frac{2}{15}$.

In the long division process detailed in Example 5, the number 15 is called the **divisor,** the number 842 is called the **dividend,** the number 56 is called the **quotient,** and the number 2 is called the **remainder.**

To check the answer obtained in a division problem, multiply the quotient by the divisor and add the remainder. The answer should be the dividend.

$$\text{(Quotient)(Divisor)} + \text{Remainder} = \text{Dividend}$$

For example, we can check the results obtained in Example 5 as follows:

$$(56)(15) + 2 = 840 + 2 = 842$$

To divide two polynomials, we first must write each polynomial in standard form. The process then follows a pattern similar to that of Example 5. The next example illustrates the procedure.

EXAMPLE 6 **Dividing Two Polynomials**

Find the quotient and the remainder when

$$3x^3 + 4x^2 + x + 7 \quad \text{is divided by} \quad x^2 + 1$$

Solution Each polynomial is in standard form. The dividend is $3x^3 + 4x^2 + x + 7$, and the divisor is $x^2 + 1$.

REMEMBER A polynomial is in standard form when its terms are written according to descending degrees.

STEP 1: Divide the leading term of the dividend, $3x^3$, by the leading term of the divisor, x^2. Enter the result, $3x$, over the term $3x^3$, as follows:

$$\begin{array}{r}
3x \\
x^2 + 1\overline{)3x^3 + 4x^2 + x + 7}
\end{array}$$

STEP 2: Multiply $3x$ by $x^2 + 1$ and enter the result below the dividend.

$$\begin{array}{r l}
3x & \\
x^2 + 1\overline{)3x^3 + 4x^2 + x + 7} & \\
\underline{3x^3 + 3x } & \leftarrow 3x \cdot (x^2 + 1) = 3x^3 + 3x
\end{array}$$

Notice that we align the $3x$ term under the x to make the next step easier.

STEP 3: Subtract and bring down the remaining terms.

$$\begin{array}{r l}
3x & \\
x^2 + 1\overline{)3x^3 + 4x^2 + x + 7} & \\
\underline{3x^3 + 3x } & \leftarrow \text{Subtract (change the signs and add).} \\
4x^2 - 2x + 7 & \leftarrow \text{Bring down the } 4x^2 \text{ and the } 7.
\end{array}$$

STEP 4: Repeat Steps 1–3 using $4x^2 - 2x + 7$ as the dividend.

$$\begin{array}{rl}
3x \quad + 4 & \\
x^2 + 1 \overline{\smash{)}\, 3x^3 + 4x^2 + x + 7} & \\
\underline{3x^3 \qquad\quad + 3x } & \\
4x^2 - 2x + 7 & \leftarrow \text{Divide } 4x^2 \text{ by } x^2 \text{ to get } 4. \\
\underline{4x^2 \qquad\quad + 4} & \leftarrow \text{Multiply } x^2 + 1 \text{ by } 4\text{; subtract.} \\
-2x + 3 &
\end{array}$$

Since x^2 does not divide $-2x$ evenly (that is, the result is not a monomial), the process ends. The quotient is $3x + 4$, and the remainder is $-2x + 3$.

Check: (Quotient)(Divisor) + Remainder

$$= (3x + 4)(x^2 + 1) + (-2x + 3)$$
$$= 3x^3 + 3x + 4x^2 + 4 + (-2x + 3)$$
$$= 3x^3 + 4x^2 + x + 7 = \text{Dividend}$$

Then

$$\frac{3x^3 + 4x^2 + x + 7}{x^2 + 1} = 3x + 4 + \frac{-2x + 3}{x^2 + 1}$$

The next example combines the steps involved in long division.

EXAMPLE 7 Dividing Two Polynomials

Find the quotient and the remainder when

$$x^4 - 3x^3 + 2x - 5 \quad \text{is divided by} \quad x^2 - x + 1$$

Solution In setting up this division problem, it is necessary to leave a space for the missing x^2 term in the dividend.

$$\begin{array}{lrl}
 & x^2 - 2x - 3 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow & x^2 - x + 1 \overline{\smash{)}\, x^4 - 3x^3 \qquad\quad + 2x - 5} & \leftarrow \text{Dividend} \\
\text{Subtract} \rightarrow & \underline{x^4 - x^3 + x^2 } & \\
 & -2x^3 - x^2 + 2x - 5 & \\
\text{Subtract} \rightarrow & \underline{-2x^3 + 2x^2 - 2x } & \\
 & -3x^2 + 4x - 5 & \\
\text{Subtract} \rightarrow & \underline{-3x^2 + 3x - 3} & \\
 & x - 2 & \leftarrow \text{Remainder}
\end{array}$$

Check: (Quotient)(Divisor) + Remainder

$$= (x^2 - 2x - 3)(x^2 - x + 1) + x - 2$$
$$= x^4 - x^3 + x^2 - 2x^3 + 2x^2 - 2x - 3x^2 + 3x - 3 + x - 2$$
$$= x^4 - 3x^3 + 2x - 5 = \text{Dividend}$$

As a result,

$$\frac{x^4 - 3x^3 + 2x - 5}{x^2 - x + 1} = x^2 - 2x - 3 + \frac{x - 2}{x^2 - x + 1}$$

The process of dividing two polynomials leads to the following result:

THEOREM

Let Q be a polynomial of positive degree and let P be a polynomial whose degree is greater than or equal to the degree of Q. The remainder after dividing P by Q is either the zero polynomial or a polynomial whose degree is less than the degree of the divisor Q.

Now Work PROBLEM 61

5 Factor Polynomials

Consider the following product:

$$(2x + 3)(x - 4) = 2x^2 - 5x - 12$$

The two polynomials on the left side are called **factors** of the polynomial on the right side. Expressing a given polynomial as a product of other polynomials, that is, finding the factors of a polynomial, is called **factoring.**

We shall restrict our discussion here to factoring polynomials in one variable into products of polynomials in one variable, where all coefficients are integers. We call this **factoring over the integers.**

Any polynomial can be written as the product of 1 times itself or as -1 times its additive inverse. If a polynomial cannot be written as the product of two other polynomials (excluding 1 and -1), then the polynomial is said to be **prime.** When a polynomial has been written as a product consisting only of prime factors, it is said to be **factored completely.** Examples of prime polynomials (over the integers) are

$$2,\quad 3,\quad 5,\quad x,\quad x + 1,\quad x - 1,\quad 3x + 4\quad x^2 + 4$$

COMMENT Over the real numbers, $3x + 4$ factors into $3(x + \frac{4}{3})$. It is the noninteger $\frac{4}{3}$ that causes $3x + 4$ to be prime over the integers.

The first factor to look for in a factoring problem is a common monomial factor present in each term of the polynomial. If one is present, use the Distributive Property to factor it out.

EXAMPLE 8 Identifying Common Monomial Factors

Polynomial	Common Monomial Factor	Remaining Factor	Factored Form
$2x + 4$	2	$x + 2$	$2x + 4 = 2(x + 2)$
$3x - 6$	3	$x - 2$	$3x - 6 = 3(x - 2)$
$2x^2 - 4x + 8$	2	$x^2 - 2x + 4$	$2x^2 - 4x + 8 = 2(x^2 - 2x + 4)$
$8x - 12$	4	$2x - 3$	$8x - 12 = 4(2x - 3)$
$x^2 + x$	x	$x + 1$	$x^2 + x = x(x + 1)$
$x^3 - 3x^2$	x^2	$x - 3$	$x^3 - 3x^2 = x^2(x - 3)$
$6x^2 + 9x$	$3x$	$2x + 3$	$6x^2 + 9x = 3x(2x + 3)$

Notice that, once all common monomial factors have been removed from a polynomial, the remaining factor is either a prime polynomial of degree 1 or a polynomial of degree 2 or higher. (Do you see why?)

The list of special products (2) through (6) given earlier provides a list of factoring formulas when the equations are read from right to left. For example, equation (2) states that if the polynomial is the difference of two squares, $x^2 - a^2$, it can be factored into $(x - a)(x + a)$. The following example illustrates several factoring techniques.

EXAMPLE 9 **Factoring Polynomials**

Factor completely each polynomial.

(a) $x^4 - 16$ (b) $x^3 - 1$ (c) $9x^2 - 6x + 1$
(d) $x^2 + 4x - 12$ (e) $3x^2 + 10x - 8$ (f) $x^3 - 4x^2 + 2x - 8$

Solution

(a) $x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$

↑ Difference of squares ↑ Difference of squares

(b) $x^3 - 1 = (x - 1)(x^2 + x + 1)$

↑ Difference of cubes

(c) $9x^2 - 6x + 1 = (3x - 1)^2$

↑ Perfect square

(d) $x^2 + 4x - 12 = (x + 6)(x - 2)$

↑ The product of 6 and −2 is −12, and the sum of 6 and −2 is 4.

$12x - 2x = 10x$

(e) $3x^2 + 10x - 8 = (3x - 2)(x + 4)$

$3x^2$ -8

(f) $x^3 - 4x^2 + 2x - 8 = (x^3 - 4x^2) + (2x - 8)$

↑ Group terms

$= x^2(x - 4) + 2(x - 4) = (x^2 + 2)(x - 4)$

↑ Distributive Property ↑ Distributive Property

COMMENT The technique used in part (f) is called **factoring by grouping.**

Now Work PROBLEMS 81, 97, AND 131

6 Complete the Square

The idea behind completing the square in one variable is to "adjust" an expression of the form $x^2 + bx$ to make it a perfect square. Perfect squares are trinomials of the form

$$x^2 + 2ax + a^2 = (x + a)^2 \text{ or } x^2 - 2ax + a^2 = (x - a)^2$$

For example, $x^2 + 6x + 9$ is a perfect square because $x^2 + 6x + 9 = (x + 3)^2$. And $p^2 - 12p + 36$ is a perfect square because $p^2 - 12p + 36 = (p - 6)^2$.

So how do we "adjust" $x^2 + bx$ to make it a perfect square? We do it by adding a number. For example, to make $x^2 + 6x$ a perfect square, add 9. But how do we know to add 9? If we divide the coefficient on the first-degree term, 6, by 2, and then square the result, we obtain 9. This approach works in general.

Completing the Square

Identify the coefficient of the first-degree term. Multiply this coefficient by $\frac{1}{2}$ and then square the result. That is, determine the value of b in $x^2 + bx$ and compute $\left(\frac{1}{2}b\right)^2$.

EXAMPLE 10 **Completing the Square**

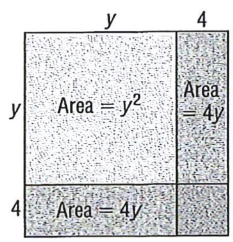
Determine the number that must be added to each expression to complete the square. Then factor the expression.

Start	Add	Result	Factored Form
$y^2 + 8y$	$\left(\frac{1}{2}\cdot 8\right)^2 = 16$	$y^2 + 8y + 16$	$(y+4)^2$
$x^2 + 12x$	$\left(\frac{1}{2}\cdot 12\right)^2 = 36$	$x^2 + 12x + 36$	$(x+6)^2$
$a^2 - 20a$	$\left(\frac{1}{2}\cdot(-20)\right)^2 = 100$	$a^2 - 20a + 100$	$(a-10)^2$
$p^2 - 5p$	$\left(\frac{1}{2}\cdot(-5)\right)^2 = \frac{25}{4}$	$p^2 - 5p + \frac{25}{4}$	$\left(p - \frac{5}{2}\right)^2$

Notice that the factored form of a perfect square is either

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2 \text{ or } x^2 - bx + \left(\frac{b}{2}\right)^2 = \left(x - \frac{b}{2}\right)^2$$

Figure 22



Now Work PROBLEM 121

Are you wondering why we call making an expression a perfect square "completing the square"? Look at the square in Figure 22. Its area is $(y + 4)^2$. The yellow area is y^2 and each orange area is $4y$ (for a total area of $8y$). The sum of these areas is $y^2 + 8y$. To complete the square, we need to add the area of the green region: $4 \cdot 4 = 16$. As a result, $y^2 + 8y + 16 = (y + 4)^2$.

A.3 Assess Your Understanding

Concepts and Vocabulary

1. The polynomial $3x^4 - 2x^3 + 13x^2 - 5$ is of degree ____. The leading coefficient is ____.
2. $(x^2 - 4)(x^2 + 4) =$ ________.
3. $(x - 2)(x^2 + 2x + 4) =$ ________.
4. ***True or False*** $4x^{-2}$ is a monomial of degree -2.
5. ***True or False*** $(x + a)(x^2 + ax + a) = x^3 + a^3$.
6. To check division, the expression that is being divided, the dividend, should equal the product of the ________ and the ________ plus the ________.
7. If factored completely, $3x^3 - 12x =$ ________.
8. To complete the square of the expression $x^2 + 5x$, you would ____ the number ____.
9. ***True or False*** The polynomial $x^2 + 4$ is prime.
10. ***True or False*** $3x^3 - 2x^2 - 6x + 4 = (3x - 2)(x^2 + 2)$.

Skill Building

In Problems 11–20, tell whether the expression is a monomial. If it is, name the variable(s) and the coefficient and give the degree of the monomial. If it is not a monomial, state why not.

11. $2x^3$
12. $-4x^2$
13. $\frac{8}{x}$
14. $-2x^{-3}$
15. $-2x^3 + 5x^2$
16. $6x^5 - 8x^2$
17. $\frac{8x}{x^2 - 1}$
18. $-\frac{2x^2}{x^3 + 1}$
19. $x^2 + 2x - 5$
20. $3x^2 + 4$

In Problems 21–30, tell whether the expression is a polynomial. If it is, give its degree. If it is not, state why not.

21. $3x^2 - 5$
22. $1 - 4x$
23. 5
24. $-\pi$
25. $3x^2 - \frac{5}{x}$
26. $\frac{3}{x} + 2$
27. $2y^3 - \sqrt{2}$
28. $10z^2 + z$
29. $\frac{x^2 + 5}{x^3 - 1}$
30. $\frac{3x^3 + 2x - 1}{x^2 + x + 1}$

In Problems 31–56, add, subtract, or multiply, as indicated. Express your answer as a single polynomial in standard form.

31. $(x^2 + 4x + 5) + (3x - 3)$

32. $(x^3 + 3x^2 + 2) + (x^2 - 4x + 4)$

33. $(x^3 - 2x^2 + 5x + 10) - (2x^2 - 4x + 3)$

34. $(x^2 - 3x - 4) - (x^3 - 3x^2 + x + 5)$

35. $6(x^3 + x^2 - 3) - 4(2x^3 - 3x^2)$

36. $8(4x^3 - 3x^2 - 1) - 6(4x^3 + 8x - 2)$

37. $9(y^2 - 3y + 4) - 6(1 - y^2)$

38. $8(1 - y^3) + 4(1 + y + y^2 + y^3)$

39. $x(x^2 + x - 4)$

40. $4x^2(x^3 - x + 2)$

41. $(x + 2)(x + 4)$

42. $(x + 3)(x + 5)$

43. $(2x + 5)(x + 2)$

44. $(3x + 1)(2x + 1)$

45. $(x - 7)(x + 7)$

46. $(x - 1)(x + 1)$

47. $(2x + 3)(2x - 3)$

48. $(3x + 2)(3x - 2)$

49. $(x + 4)^2$

50. $(x - 5)^2$

51. $(2x - 3)^2$

52. $(3x - 4)^2$

53. $(x - 2)^3$

54. $(x + 1)^3$

55. $(2x + 1)^3$

56. $(3x - 2)^3$

In Problems 57–72, find the quotient and the remainder. Check your work by verifying that

$$(\textit{Quotient})(\textit{Divisor}) + \textit{Remainder} = \textit{Dividend}$$

57. $4x^3 - 3x^2 + x + 1$ divided by $x + 2$

58. $3x^3 - x^2 + x - 2$ divided by $x + 2$

59. $4x^3 - 3x^2 + x + 1$ divided by x^2

60. $3x^3 - x^2 + x - 2$ divided by x^2

61. $5x^4 - 3x^2 + x + 1$ divided by $x^2 + 2$

62. $5x^4 - x^2 + x - 2$ divided by $x^2 + 2$

63. $4x^5 - 3x^2 + x + 1$ divided by $2x^3 - 1$

64. $3x^5 - x^2 + x - 2$ divided by $3x^3 - 1$

65. $2x^4 - 3x^3 + x + 1$ divided by $2x^2 + x + 1$

66. $3x^4 - x^3 + x - 2$ divided by $3x^2 + x + 1$

67. $-4x^3 + x^2 - 4$ divided by $x - 1$

68. $-3x^4 - 2x - 1$ divided by $x - 1$

69. $1 - x^2 + x^4$ divided by $x^2 + x + 1$

70. $1 - x^2 + x^4$ divided by $x^2 - x + 1$

71. $x^3 - a^3$ divided by $x - a$

72. $x^5 - a^5$ divided by $x - a$

In Problems 73–120, factor completely each polynomial. If the polynomial cannot be factored, say it is prime.

73. $x^2 - 36$

74. $x^2 - 9$

75. $2 - 8x^2$

76. $3 - 27x^2$

77. $x^2 + 11x + 10$

78. $x^2 + 5x + 4$

79. $x^2 - 10x + 21$

80. $x^2 - 6x + 8$

81. $4x^2 - 8x + 32$

82. $3x^2 - 12x + 15$

83. $x^2 + 4x + 16$

84. $x^2 + 12x + 36$

85. $15 + 2x - x^2$

86. $14 + 6x - x^2$

87. $3x^2 - 12x - 36$

88. $x^3 + 8x^2 - 20x$

89. $y^4 + 11y^3 + 30y^2$

90. $3y^3 - 18y^2 - 48y$

91. $4x^2 + 12x + 9$

92. $9x^2 - 12x + 4$

93. $6x^2 + 8x + 2$

94. $8x^2 + 6x - 2$

95. $x^4 - 81$

96. $x^4 - 1$

97. $x^6 - 2x^3 + 1$

98. $x^6 + 2x^3 + 1$

99. $x^7 - x^5$

100. $x^8 - x^5$

101. $16x^2 + 24x + 9$

102. $9x^2 - 24x + 16$

103. $5 + 16x - 16x^2$

104. $5 + 11x - 16x^2$

105. $4y^2 - 16y + 15$

106. $9y^2 + 9y - 4$

107. $1 - 8x^2 - 9x^4$

108. $4 - 14x^2 - 8x^4$

109. $x(x + 3) - 6(x + 3)$

110. $5(3x - 7) + x(3x - 7)$

111. $(x + 2)^2 - 5(x + 2)$

112. $(x - 1)^2 - 2(x - 1)$

113. $(3x-2)^3-27$

114. $(5x+1)^3-1$

115. $3(x^2+10x+25)-4(x+5)$

116. $7(x^2-6x+9)+5(x-3)$

117. x^3+2x^2-x-2

118. x^3-3x^2-x+3

119. x^4-x^3+x-1

120. x^4+x^3+x+1

In Problems 121–126, determine the number that should be added to complete the square of each expression. Then factor each expression.

121. x^2+10x

122. p^2+14p

123. y^2-6y

124. x^2-4x

125. $x^2-\frac{1}{2}x$

126. $x^2+\frac{1}{3}x$

Applications and Extensions

In Problems 127–136, expressions that occur in calculus are given. Factor completely each expression.

127. $2(3x+4)^2+(2x+3)\cdot 2(3x+4)\cdot 3$

128. $5(2x+1)^2+(5x-6)\cdot 2(2x+1)\cdot 2$

129. $2x(2x+5)+x^2\cdot 2$

130. $3x^2(8x-3)+x^3\cdot 8$

131. $2(x+3)(x-2)^3+(x+3)^2\cdot 3(x-2)^2$

132. $4(x+5)^3(x-1)^2+(x+5)^4\cdot 2(x-1)$

133. $(4x-3)^2+x\cdot 2(4x-3)\cdot 4$

134. $3x^2(3x+4)^2+x^3\cdot 2(3x+4)\cdot 3$

135. $2(3x-5)\cdot 3(2x+1)^3+(3x-5)^2\cdot 3(2x+1)^2\cdot 2$

136. $3(4x+5)^2\cdot 4(5x+1)^2+(4x+5)^3\cdot 2(5x+1)\cdot 5$

137. Show that x^2+4 is prime.

138. Show that x^2+x+1 is prime.

Explaining Concepts: Discussion and Writing

139. Explain why the degree of the product of two nonzero polynomials equals the sum of their degrees.

140. Explain why the degree of the sum of two polynomials of different degrees equals the larger of their degrees.

141. Give a careful statement about the degree of the sum of two polynomials of the same degree.

142. Do you prefer to memorize the rule for the square of a binomial $(x+a)^2$ or to use FOIL to obtain the product? Write a brief position paper defending your choice.

143. Make up a polynomial that factors into a perfect square.

144. Explain to a fellow student what you look for first when presented with a factoring problem. What do you do next?

A.4 Synthetic Division

OBJECTIVE 1 Divide Polynomials Using Synthetic Division (p. A32)

1 Divide Polynomials Using Synthetic Division

To find the quotient as well as the remainder when a polynomial of degree 1 or higher is divided by $x-c$, a shortened version of long division, called **synthetic division,** makes the task simpler.

To see how synthetic division works, we will use long division to divide the polynomial $2x^3-x^2+3$ by $x-3$.

$$\begin{array}{r l}
 & 2x^2 + 5x + 15 \quad \leftarrow \text{Quotient} \\
x-3 \,) & 2x^3 - x^2 \qquad + 3 \\
 & \underline{2x^3 - 6x^2} \\
 & \qquad 5x^2 \\
 & \qquad \underline{5x^2 - 15x} \\
 & \qquad\qquad 15x + 3 \\
 & \qquad\qquad \underline{15x - 45} \\
 & \qquad\qquad\qquad 48 \quad \leftarrow \text{Remainder}
\end{array}$$

Check: (Divisor) · (Quotient) + Remainder

$$= (x-3)(2x^2+5x+15)+48$$
$$= 2x^3+5x^2+15x-6x^2-15x-45+48$$
$$= 2x^3-x^2+3$$

The process of synthetic division arises from rewriting the long division in a more compact form, using simpler notation. For example, in the long division on the previous page, the terms in blue are not really necessary because they are identical to the terms directly above them. With these terms removed, we have

$$\begin{array}{r|rrrr}
 & 2x^2 & +5x & +15 & \\ \hline
x-3 & 2x^3 & -x^2 & & +3 \\
 & & -6x^2 & & \\ \cline{2-3}
 & & 5x^2 & & \\
 & & & -15x & \\ \cline{3-5}
 & & & 15x & \\
 & & & & -45 \\ \cline{4-5}
 & & & & 48
\end{array}$$

Most of the x's that appear in this process can also be removed, provided that we are careful about positioning each coefficient. In this regard, we will need to use 0 as the coefficient of x in the dividend, because that power of x is missing. Now we have

$$\begin{array}{r|rrrr}
 & 2x^2 & +5x & +15 & \\ \hline
x-3 & 2 & -1 & 0 & 3 \\
 & & -6 & & \\ \cline{2-3}
 & & 5 & & \\
 & & & -15 & \\ \cline{3-5}
 & & & 15 & \\
 & & & & -45 \\ \cline{4-5}
 & & & & 48
\end{array}$$

We can make this display more compact by moving the lines up until the numbers in blue align horizontally.

$$\begin{array}{r|rrrrl}
 & 2x^2 & +5x & +15 & & \text{Row 1} \\ \hline
x-3 & 2 & -1 & 0 & 3 & \text{Row 2} \\
 & & -6 & -15 & -45 & \text{Row 3} \\ \cline{2-5}
 & \bigcirc & 5 & 15 & 48 & \text{Row 4}
\end{array}$$

Because the leading coefficient of the divisor is always 1, we know that the leading coefficient of the dividend will also be the leading coefficient of the quotient. So we place the leading coefficient of the dividend, 2, in the circled position. Now, the first three numbers in row 4 are precisely the coefficients of the quotient, and the last number in row 4 is the remainder. Thus, row 1 is not really needed, so we can compress the process to three rows, where the bottom row contains both the coefficients of the quotient and the remainder.

$$\begin{array}{r|rrrrl}
x-3 & 2 & -1 & 0 & 3 & \text{Row 1} \\
 & & -6 & -15 & -45 & \text{Row 2 (subtract)} \\ \cline{2-5}
 & 2 & 5 & 15 & 48 & \text{Row 3}
\end{array}$$

Recall that the entries in row 3 are obtained by subtracting the entries in row 2 from those in row 1. Rather than subtracting the entries in row 2, we can change the sign of each entry and add. With this modification, our display will look like this:

$$\begin{array}{r|rrrrl}
x-3 & 2 & -1 & 0 & 3 & \text{Row 1} \\
 & & 6 & 15 & 45 & \text{Row 2 (add)} \\ \cline{2-5}
 & 2 & 5 & 15 & 48 & \text{Row 3}
\end{array}$$

Notice that the entries in row 2 are three times the prior entries in row 3. Our last modification to the display replaces the $x - 3$ by 3. The entries in row 3 give the quotient and the remainder, as shown next.

$$\begin{array}{r|rrrl} 3 & 2 & -1 & 0 & 3 \quad \text{Row 1} \\ & & 6 & 15 & 45 \quad \text{Row 2 (add)} \\ \hline & 2 & 5 & 15 & 48 \quad \text{Row 3} \end{array}$$

Quotient: $2x^2 + 5x + 15$ Remainder: 48

EXAMPLE 1

Using Synthetic Division to Find the Quotient and Remainder

Use synthetic division to find the quotient and remainder when

$$x^3 - 4x^2 - 5 \quad \text{is divided by} \quad x - 3$$

Solution

Step 1: Write the dividend in descending powers of x. Then copy the coefficients, remembering to insert a 0 for any missing powers of x.

$$1 \quad -4 \quad 0 \quad -5 \qquad \text{Row 1}$$

Step 2: Insert the usual division symbol. In synthetic division, the divisor is of the form $x - c$, and c is the number placed to the left of the division symbol. Here, since the divisor is $x - 3$, we insert 3 to the left of the division symbol.

$$\begin{array}{r|rrrl} 3 & 1 & -4 & 0 & -5 \quad \text{Row 1} \end{array}$$

Step 3: Bring the 1 down two rows, and enter it in row 3.

$$\begin{array}{r|rrrl} 3 & 1 & -4 & 0 & -5 \quad \text{Row 1} \\ & \downarrow & & & \quad\;\; \text{Row 2} \\ \hline & 1 & & & \quad\;\; \text{Row 3} \end{array}$$

Step 4: Multiply the latest entry in row 3 by 3, and place the result in row 2, one column over to the right.

$$\begin{array}{r|rrrl} 3 & 1 & -4 & 0 & -5 \quad \text{Row 1} \\ & & 3 & & \quad\;\; \text{Row 2} \\ \hline & 1 & & & \quad\;\; \text{Row 3} \end{array}$$

Step 5: Add the entry in row 2 to the entry above it in row 1, and enter the sum in row 3.

$$\begin{array}{r|rrrl} 3 & 1 & -4 & 0 & -5 \quad \text{Row 1} \\ & & 3 & & \quad\;\; \text{Row 2} \\ \hline & 1 & -1 & & \quad\;\; \text{Row 3} \end{array}$$

Step 6: Repeat Steps 4 and 5 until no more entries are available in row 1.

$$\begin{array}{r|rrrl} 3 & 1 & -4 & 0 & -5 \quad \text{Row 1} \\ & & 3 & -3 & -9 \quad \text{Row 2} \\ \hline & 1 & -1 & -3 & -14 \quad \text{Row 3} \end{array}$$

Step 7: The final entry in row 3, the -14, is the remainder; the other entries in row 3, the 1, -1, and -3, are the coefficients (in descending order) of a polynomial whose degree is 1 less than that of the dividend. This is the quotient. Thus,

$$\text{Quotient} = x^2 - x - 3 \qquad \text{Remainder} = -14$$

✓**Check:** (Divisor)(Quotient) + Remainder

$$\begin{aligned} &= (x - 3)(x^2 - x - 3) + (-14) \\ &= (x^3 - x^2 - 3x - 3x^2 + 3x + 9) + (-14) \\ &= x^3 - 4x^2 - 5 = \text{Dividend} \end{aligned}$$

The next example combines all the steps.

EXAMPLE 2 **Using Synthetic Division to Verify a Factor**

Use synthetic division to show that $x + 3$ is a factor of

$$2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$$

Solution The divisor is $x + 3 = x - (-3)$, so we place -3 to the left of the division symbol. Then the row 3 entries will be multiplied by -3, entered in row 2, and added to row 1.

$$\begin{array}{r|rrrrrr l} -3 & 2 & 5 & -2 & 2 & -2 & 3 & \text{Row 1} \\ & & -6 & 3 & -3 & 3 & -3 & \text{Row 2} \\ \hline & 2 & -1 & 1 & -1 & 1 & 0 & \text{Row 3} \end{array}$$

Because the remainder is 0, we have

$$\begin{aligned} &(\text{Divisor})(\text{Quotient}) + \text{Remainder} \\ &= (x + 3)(2x^4 - x^3 + x^2 - x + 1) = 2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3 \end{aligned}$$

As we see, $x + 3$ is a factor of $2x^5 + 5x^4 - 2x^3 + 2x^2 - 2x + 3$.

As Example 2 illustrates, the remainder after division gives information about whether the divisor is, or is not, a factor. We shall have more to say about this in Chapter 4.

Now Work PROBLEMS 7 AND 17

A.4 Assess Your Understanding

Concepts and Vocabulary

1. To check division, the expression that is being divided, the dividend, should equal the product of the ________ and the ________ plus the ________.
2. To divide $2x^3 - 5x + 1$ by $x + 3$ using synthetic division, the first step is to write ____ $\overline{)\quad\quad}$ ________.
3. ***True or False*** In using synthetic division, the divisor is always a polynomial of degree 1, whose leading coefficient is 1.
4. ***True or False*** $\begin{array}{r|rrrr} -2 & 5 & 3 & 2 & 1 \\ & & -10 & 14 & -32 \\ \hline & 5 & -7 & 16 & -31 \end{array}$ means $\dfrac{5x^3 + 3x^2 + 2x + 1}{x + 2} = 5x^2 - 7x + 16 + \dfrac{-31}{x + 2}$.

Skill Building

In Problems 5–16, use synthetic division to find the quotient and remainder when:

5. $x^3 - x^2 + 2x + 4$ is divided by $x - 2$
6. $x^3 + 2x^2 - 3x + 1$ is divided by $x + 1$
7. $3x^3 + 2x^2 - x + 3$ is divided by $x - 3$
8. $-4x^3 + 2x^2 - x + 1$ is divided by $x + 2$
9. $x^5 - 4x^3 + x$ is divided by $x + 3$
10. $x^4 + x^2 + 2$ is divided by $x - 2$
11. $4x^6 - 3x^4 + x^2 + 5$ is divided by $x - 1$
12. $x^5 + 5x^3 - 10$ is divided by $x + 1$
13. $0.1x^3 + 0.2x$ is divided by $x + 1.1$
14. $0.1x^2 - 0.2$ is divided by $x + 2.1$
15. $x^5 - 1$ is divided by $x - 1$
16. $x^5 + 1$ is divided by $x + 1$

In Problems 17–26, use synthetic division to determine whether $x - c$ is a factor of the given polynomial.

17. $4x^3 - 3x^2 - 8x + 4;\ x - 2$
18. $-4x^3 + 5x^2 + 8;\ x + 3$
19. $3x^4 - 6x^3 - 5x + 10;\ x - 2$
20. $4x^4 - 15x^2 - 4;\ x - 2$
21. $3x^6 + 82x^3 + 27;\ x + 3$
22. $2x^6 - 18x^4 + x^2 - 9;\ x + 3$

23. $4x^6 - 64x^4 + x^2 - 15; \quad x + 4$

24. $x^6 - 16x^4 + x^2 - 16; \quad x + 4$

25. $2x^4 - x^3 + 2x - 1; \quad x - \frac{1}{2}$

26. $3x^4 + x^3 - 3x + 1; \quad x + \frac{1}{3}$

Applications and Extensions

27. Find the sum of $a, b, c,$ and d if

$$\frac{x^3 - 2x^2 + 3x + 5}{x + 2} = ax^2 + bx + c + \frac{d}{x + 2}$$

Explaining Concepts: Discussion and Writing

28. When dividing a polynomial by $x - c$, do you prefer to use long division or synthetic division? Does the value of c make a difference to you in choosing? Give reasons.

A.5 Rational Expressions

OBJECTIVES

1 Reduce a Rational Expression to Lowest Terms

If we form the quotient of two polynomials, the result is called a **rational expression.** Some examples of rational expressions are

(a) $\frac{x^3 + 1}{x}$ (b) $\frac{3x^2 + x - 2}{x^2 + 5}$ (c) $\frac{x}{x^2 - 1}$ (d) $\frac{xy^2}{(x - y)^2}$

Expressions (a), (b), and (c) are rational expressions in one variable, x, whereas (d) is a rational expression in two variables, x and y.

Rational expressions are described in the same manner as rational numbers. In expression (a), the polynomial $x^3 + 1$ is called the **numerator,** and x is called the **denominator.** When the numerator and denominator of a rational expression contain no common factors (except 1 and -1), we say that the rational expression is **reduced to lowest terms,** or **simplified.**

The polynomial in the denominator of a rational expression cannot be equal to 0 because division by 0 is not defined. For example, for the expression $\frac{x^3 + 1}{x}$, x cannot take on the value 0. The domain of the variable x is $\{x \mid x \neq 0\}$.

A rational expression is reduced to lowest terms by factoring completely the numerator and the denominator and canceling any common factors by using the Cancellation Property:

WARNING Apply the Cancellation Property only to rational expressions written in factored form. Be sure to cancel only common factors! ■

$$\frac{a\cancel{c}}{b\cancel{c}} = \frac{a}{b} \quad \text{if } b \neq 0, c \neq 0 \qquad (1)$$

EXAMPLE 1 **Reducing a Rational Expression to Lowest Terms**

Reduce each rational expression to lowest terms.

(a) $\frac{x^2 + 4x + 4}{x^2 + 3x + 2}$ (b) $\frac{x^3 - 8}{x^3 - 2x^2}$ (c) $\frac{8 - 2x}{x^2 - x - 12}$

Solution (a) $\dfrac{x^2+4x+4}{x^2+3x+2} = \dfrac{\cancel{(x+2)}(x+2)}{\cancel{(x+2)}(x+1)} = \dfrac{x+2}{x+1} \qquad x \neq -2, -1$

(b) $\dfrac{x^3-8}{x^3-2x^2} = \dfrac{\cancel{(x-2)}(x^2+2x+4)}{x^2\cancel{(x-2)}} = \dfrac{x^2+2x+4}{x^2} \qquad x \neq 0, 2$

(c) $\dfrac{8-2x}{x^2-x-12} = \dfrac{2(4-x)}{(x-4)(x+3)} = \dfrac{2(-1)\cancel{(x-4)}}{\cancel{(x-4)}(x+3)} = -\dfrac{2}{x+3} \qquad x \neq -3, 4$

Now Work PROBLEM 5

2 Multiply and Divide Rational Expressions

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers. If $\frac{a}{b}$ and $\frac{c}{d}$, $b \neq 0$, $d \neq 0$, are two rational expressions, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad \text{if } b \neq 0,\ d \neq 0 \tag{2}$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \quad \text{if } b \neq 0,\ c \neq 0,\ d \neq 0 \tag{3}$$

In using equations (2) and (3) with rational expressions, be sure first to factor each polynomial completely so that common factors can be canceled. Leave your answer in factored form.

EXAMPLE 2 **Multiplying and Dividing Rational Expressions**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

(a) $\dfrac{x^2-2x+1}{x^3+x} \cdot \dfrac{4x^2+4}{x^2+x-2}$ (b) $\dfrac{\frac{x+3}{x^2-4}}{\frac{x^2-x-12}{x^3-8}}$

Solution (a) $\dfrac{x^2-2x+1}{x^3+x} \cdot \dfrac{4x^2+4}{x^2+x-2} = \dfrac{(x-1)^2}{x(x^2+1)} \cdot \dfrac{4(x^2+1)}{(x+2)(x-1)}$

$$= \frac{(x-1)^{\cancel{2}}(4)\cancel{(x^2+1)}}{x\cancel{(x^2+1)}(x+2)\cancel{(x-1)}}$$

$$= \frac{4(x-1)}{x(x+2)} \qquad x \neq -2, 0, 1$$

$$\text{(b)}\ \frac{\dfrac{x+3}{x^2-4}}{\dfrac{x^2-x-12}{x^3-8}} = \frac{x+3}{x^2-4}\cdot\frac{x^3-8}{x^2-x-12}$$

$$= \frac{x+3}{(x-2)(x+2)}\cdot\frac{(x-2)(x^2+2x+4)}{(x-4)(x+3)}$$

$$= \frac{\cancel{(x+3)}\,\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}(x+2)(x-4)\cancel{(x+3)}}$$

$$= \frac{x^2+2x+4}{(x+2)(x-4)} \qquad x \neq -3, -2, 2, 4$$

Now Work PROBLEM 13

3 Add and Subtract Rational Expressions

The rules for adding and subtracting rational expressions are the same as the rules for adding and subtracting rational numbers. So, if the denominators of two rational expressions to be added (or subtracted) are equal, we add (or subtract) the numerators and keep the common denominator.

In Words

To add (or subtract) two rational expressions with the same denominator, keep the common denominator and add (or subtract) the numerators.

If $\frac{a}{b}$ and $\frac{c}{b}$ are two rational expressions, then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b} \qquad \frac{a}{b}-\frac{c}{b}=\frac{a-c}{b} \qquad \text{if } b \neq 0 \qquad \textbf{(4)}$$

EXAMPLE 3 **Adding Rational Expressions with Equal Denominators**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{2x^2-4}{2x+5}+\frac{x+3}{2x+5} \qquad x \neq -\frac{5}{2}$$

Solution

$$\frac{2x^2-4}{2x+5}+\frac{x+3}{2x+5}=\frac{(2x^2-4)+(x+3)}{2x+5}$$

$$=\frac{2x^2+x-1}{2x+5}=\frac{(2x-1)(x+1)}{2x+5}$$

Now Work PROBLEM 21

If the denominators of two rational expressions to be added or subtracted are not equal, we can use the general formulas for adding and subtracting quotients.

$$\frac{a}{b}+\frac{c}{d}=\frac{a\cdot d}{b\cdot d}+\frac{b\cdot c}{b\cdot d}=\frac{ad+bc}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(5a)}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{a\cdot d}{b\cdot d}-\frac{b\cdot c}{b\cdot d}=\frac{ad-bc}{bd} \qquad \text{if } b \neq 0, d \neq 0 \qquad \textbf{(5b)}$$

EXAMPLE 4

Subtracting Rational Expressions with Unequal Denominators

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{x^2}{x^2-4} - \frac{1}{x} \qquad x \neq -2, 0, 2$$

Solution

$$\frac{x^2}{x^2-4} - \frac{1}{x} \underset{\uparrow \, (5b)}{=} \frac{x^2}{x^2-4}\cdot\frac{x}{x} - \frac{x^2-4}{x^2-4}\cdot\frac{1}{x} = \frac{x^2(x) - (x^2-4)(1)}{(x^2-4)(x)}$$

$$= \frac{x^3 - x^2 + 4}{(x-2)(x+2)(x)}$$

Now Work PROBLEM 23

4 Use the Least Common Multiple Method

If the denominators of two rational expressions to be added (or subtracted) have common factors, we usually do not use the general rules given by equations (5a) and (5b). Just as with fractions, we apply the **least common multiple (LCM) method.** The LCM method uses the polynomial of least degree that has each denominator polynomial as a factor.

The LCM Method for Adding or Subtracting Rational Expressions

The Least Common Multiple (LCM) Method requires four steps:

STEP 1: Factor completely the polynomial in the denominator of each rational expression.

STEP 2: The LCM of the denominator is the product of each of these factors raised to a power equal to the greatest number of times that the factor occurs in the polynomials.

STEP 3: Write each rational expression using the LCM as the common denominator.

STEP 4: Add or subtract the rational expressions using equation (4).

We begin with an example that goes through Steps 1 and 2.

EXAMPLE 5

Finding the Least Common Multiple

Find the least common multiple of the following pair of polynomials:

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

Solution

STEP 1: The polynomials are already factored completely as

$$x(x-1)^2(x+1) \quad \text{and} \quad 4(x-1)(x+1)^3$$

STEP 2: Start by writing the factors of the left-hand polynomial. (Or you could start with the one on the right.)

$$x(x-1)^2(x+1)$$

Now look at the right-hand polynomial. Its first factor, 4, does not appear in our list, so we insert it.

$$4x(x-1)^2(x+1)$$

The next factor, $x - 1$, is already in our list, so no change is necessary. The final factor is $(x + 1)^3$. Since our list has $x + 1$ to the first power only, we replace $x + 1$ in the list by $(x + 1)^3$. The LCM is

$$4x(x - 1)^2(x + 1)^3$$

Notice that the LCM is, in fact, the polynomial of least degree that contains $x(x - 1)^2(x + 1)$ and $4(x - 1)(x + 1)^3$ as factors.

EXAMPLE 6 **Using the Least Common Multiple to Add Rational Expressions**

Perform the indicated operation and simplify the result. Leave your answer in factored form.

$$\frac{x}{x^2 + 3x + 2} + \frac{2x - 3}{x^2 - 1} \qquad x \neq -2, -1, 1$$

Solution **STEP 1:** Factor completely the polynomials in the denominators.

$$x^2 + 3x + 2 = (x + 2)(x + 1)$$
$$x^2 - 1 = (x - 1)(x + 1)$$

STEP 2: The LCM is $(x + 2)(x + 1)(x - 1)$. Do you see why?

STEP 3: Write each rational expression using the LCM as the denominator.

$$\frac{x}{x^2 + 3x + 2} = \frac{x}{(x + 2)(x + 1)} = \frac{x}{(x + 2)(x + 1)} \cdot \frac{x - 1}{x - 1} = \frac{x(x - 1)}{(x + 2)(x + 1)(x - 1)}$$

↑ Multiply numerator and denominator by $x - 1$ to get the LCM in the denominator.

$$\frac{2x - 3}{x^2 - 1} = \frac{2x - 3}{(x - 1)(x + 1)} = \frac{2x - 3}{(x - 1)(x + 1)} \cdot \frac{x + 2}{x + 2} = \frac{(2x - 3)(x + 2)}{(x - 1)(x + 1)(x + 2)}$$

↑ Multiply numerator and denominator by $x + 2$ to get the LCM in the denominator.

STEP 4: Now add by using equation (4).

$$\frac{x}{x^2 + 3x + 2} + \frac{2x - 3}{x^2 - 1} = \frac{x(x - 1)}{(x + 2)(x + 1)(x - 1)} + \frac{(2x - 3)(x + 2)}{(x + 2)(x + 1)(x - 1)}$$

$$= \frac{(x^2 - x) + (2x^2 + x - 6)}{(x + 2)(x + 1)(x - 1)}$$

$$= \frac{3x^2 - 6}{(x + 2)(x + 1)(x - 1)} = \frac{3(x^2 - 2)}{(x + 2)(x + 1)(x - 1)}$$

Now Work PROBLEM 27

5 Simplify Complex Rational Expressions

When sums and/or differences of rational expressions appear as the numerator and/or denominator of a quotient, the quotient is called a **complex rational expression.*** For example,

$$\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \quad \text{and} \quad \frac{\frac{x^2}{x^2 - 4} - 3}{\frac{x - 3}{x + 2} - 1}$$

* Some texts use the term **complex fraction.**

are complex rational expressions. To **simplify** a complex rational expression means to write it as a rational expression reduced to lowest terms. This can be accomplished in either of two ways.

Simplifying a Complex Rational Expression

METHOD 1: Treat the numerator and denominator of the complex rational expression separately, performing whatever operations are indicated and simplifying the results. Follow this by simplifying the resulting rational expression.

METHOD 2: Find the LCM of the denominators of all rational expressions that appear in the complex rational expression. Multiply the numerator and denominator of the complex rational expression by the LCM and simplify the result.

We use both methods in the next example. By carefully studying each method, you can discover situations in which one method may be easier to use than the other.

EXAMPLE 7 Simplifying a Complex Rational Expression

Simplify: $\dfrac{\dfrac{1}{2}+\dfrac{3}{x}}{\dfrac{x+3}{4}}$ $\quad x \neq -3, 0$

Solution *Method 1:* First, perform the indicated operation in the numerator, and then divide.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{\frac{1\cdot x+2\cdot 3}{2\cdot x}}{\frac{x+3}{4}} = \frac{\frac{x+6}{2x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{x+6}{2x}\cdot\frac{4}{x+3}$$

Rule for adding quotients; Rule for dividing quotients

$$\underset{\uparrow}{=} \frac{(x+6)\cdot 4}{2\cdot x\cdot(x+3)} = \frac{\cancel{2}\cdot 2\cdot(x+6)}{\cancel{2}\cdot x\cdot(x+3)} = \frac{2(x+6)}{x(x+3)}$$

Rule for multiplying quotients

Method 2: The rational expressions that appear in the complex rational expression are

$$\frac{1}{2},\ \frac{3}{x},\ \frac{x+3}{4}$$

The LCM of their denominators is $4x$. Multiply the numerator and denominator of the complex rational expression by $4x$ and then simplify.

$$\frac{\frac{1}{2}+\frac{3}{x}}{\frac{x+3}{4}} \underset{\uparrow}{=} \frac{4x\cdot\left(\frac{1}{2}+\frac{3}{x}\right)}{4x\cdot\left(\frac{x+3}{4}\right)} \underset{\uparrow}{=} \frac{4x\cdot\frac{1}{2}+4x\cdot\frac{3}{x}}{\frac{4x\cdot(x+3)}{4}}$$

Multiply the numerator and denominator by $4x$.; Use the Distributive Property in the numerator.

$$= \frac{2\cdot 2x\cdot\frac{1}{\cancel{2}}+4\cancel{x}\cdot\frac{3}{\cancel{x}}}{\frac{\cancel{4}x\cdot(x+3)}{\cancel{4}}} \underset{\uparrow}{=} \frac{2x+12}{x(x+3)} \underset{\uparrow}{=} \frac{2(x+6)}{x(x+3)}$$

Simplify.; Factor.

EXAMPLE 8 **Simplifying a Complex Rational Expression**

Simplify: $\dfrac{\dfrac{x^2}{x-4}+2}{\dfrac{2x-2}{x}-1}$ $\quad x \neq 0, 2, 4$

Solution We will use Method 1.

$$\frac{\dfrac{x^2}{x-4}+2}{\dfrac{2x-2}{x}-1} = \frac{\dfrac{x^2}{x-4}+\dfrac{2(x-4)}{x-4}}{\dfrac{2x-2}{x}-\dfrac{x}{x}} = \frac{\dfrac{x^2+2x-8}{x-4}}{\dfrac{2x-2-x}{x}}$$

$$= \frac{\dfrac{(x+4)(x-2)}{x-4}}{\dfrac{x-2}{x}} = \frac{(x+4)\cancel{(x-2)}}{x-4}\cdot\frac{x}{\cancel{x-2}}$$

$$= \frac{(x+4)\cdot x}{x-4}$$

Now Work PROBLEM 31

A.5 Assess Your Understanding

Concepts and Vocabulary

1. When the numerator and denominator of a rational expression contain no common factors (except 1 and −1), the rational expression is in ________ ________.
2. LCM is an abbreviation for ________ ________ ________.
3. ***True or False*** The rational expression $\dfrac{2x^3-4x}{x-2}$ is reduced to lowest terms.
4. ***True or False*** The LCM of $2x^3+6x^2$ and $6x^4+4x^3$ is $4x^3(x+1)$.

Skill Building

In Problems 5–12, reduce each rational expression to lowest terms.

5. $\dfrac{3x+9}{x^2-9}$
6. $\dfrac{4x^2+8x}{12x+24}$
7. $\dfrac{x^2-2x}{3x-6}$
8. $\dfrac{15x^2+24x}{3x^2}$
9. $\dfrac{24x^2}{12x^2-6x}$
10. $\dfrac{x^2+4x+4}{x^2-4}$
11. $\dfrac{y^2-25}{2y^2-8y-10}$
12. $\dfrac{3y^2-y-2}{3y^2+5y+2}$

In Problems 13–34, perform the indicated operation and simplify the result. Leave your answer in factored form.

13. $\dfrac{3x+6}{5x^2}\cdot\dfrac{x}{x^2-4}$
14. $\dfrac{3}{2x}\cdot\dfrac{x^2}{6x+10}$
15. $\dfrac{4x^2}{x^2-16}\cdot\dfrac{x^3-64}{2x}$
16. $\dfrac{12}{x^2+x}\cdot\dfrac{x^3+1}{4x-2}$
17. $\dfrac{\dfrac{8x}{x^2-1}}{\dfrac{10x}{x+1}}$
18. $\dfrac{\dfrac{x-2}{4x}}{\dfrac{x^2-4x+4}{12x}}$
19. $\dfrac{\dfrac{4-x}{4+x}}{\dfrac{4x}{x^2-16}}$
20. $\dfrac{\dfrac{3+x}{3-x}}{\dfrac{x^2-9}{9x^3}}$
21. $\dfrac{x^2}{2x-3}-\dfrac{4}{2x-3}$
22. $\dfrac{3x^2}{2x-1}-\dfrac{9}{2x-1}$
23. $\dfrac{x}{x^2-4}+\dfrac{1}{x}$
24. $\dfrac{x-1}{x^3}+\dfrac{x}{x^2+1}$

25. $\dfrac{x}{x^2-7x+6}-\dfrac{x}{x^2-2x-24}$

26. $\dfrac{x}{x-3}-\dfrac{x+1}{x^2+5x-24}$

27. $\dfrac{4x}{x^2-4}-\dfrac{2}{x^2+x-6}$

28. $\dfrac{3x}{x-1}-\dfrac{x-4}{x^2-2x+1}$

29. $\dfrac{3}{(x-1)^2(x+1)}+\dfrac{2}{(x-1)(x+1)^2}$

30. $\dfrac{2}{(x+2)^2(x-1)}-\dfrac{6}{(x+2)(x-1)^2}$

31. $\dfrac{1+\dfrac{1}{x}}{1-\dfrac{1}{x}}$

32. $\dfrac{4+\dfrac{1}{x^2}}{3-\dfrac{1}{x^2}}$

33. $\dfrac{\dfrac{x-2}{x+2}+\dfrac{x-1}{x+1}}{\dfrac{x}{x+1}-\dfrac{2x-3}{x}}$

34. $\dfrac{\dfrac{2x+5}{x}-\dfrac{x}{x-3}}{\dfrac{x^2}{x-3}-\dfrac{(x+1)^2}{x+3}}$

Applications and Extensions

In Problems 35–42, expressions that occur in calculus are given. Reduce each expression to lowest terms.

35. $\dfrac{(2x+3)\cdot 3-(3x-5)\cdot 2}{(3x-5)^2}$

36. $\dfrac{(4x+1)\cdot 5-(5x-2)\cdot 4}{(5x-2)^2}$

37. $\dfrac{x\cdot 2x-(x^2+1)\cdot 1}{(x^2+1)^2}$

38. $\dfrac{x\cdot 2x-(x^2-4)\cdot 1}{(x^2-4)^2}$

39. $\dfrac{(3x+1)\cdot 2x-x^2\cdot 3}{(3x+1)^2}$

40. $\dfrac{(2x-5)\cdot 3x^2-x^3\cdot 2}{(2x-5)^2}$

41. $\dfrac{(x^2+1)\cdot 3-(3x+4)\cdot 2x}{(x^2+1)^2}$

42. $\dfrac{(x^2+9)\cdot 2-(2x-5)\cdot 2x}{(x^2+9)^2}$

43. The Lensmaker's Equation The focal length f of a lens with index of refraction n is

$$\frac{1}{f}=(n-1)\left[\frac{1}{R_1}+\frac{1}{R_2}\right]$$

where R_1 and R_2 are the radii of curvature of the front and back surfaces of the lens. Express f as a rational expression. Evaluate the rational expression for $n=1.5$, $R_1=0.1$ meter, and $R_2=0.2$ meter.

44. Electrical Circuits An electrical circuit contains three resistors connected in parallel. If the resistance of each is R_1, R_2, and R_3 ohms, respectively, their combined resistance R is given by the formula

$$\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}$$

Express R as a rational expression. Evaluate R for $R_1=5$ ohms, $R_2=4$ ohms, and $R_3=10$ ohms.

Explaining Concepts: Discussion and Writing

45. The following expressions are called **continued fractions:**

$$1+\frac{1}{x},\quad 1+\cfrac{1}{1+\cfrac{1}{x}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}},\quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{x}}}},\quad \ldots$$

Each simplifies to an expression of the form

$$\frac{ax+b}{bx+c}$$

Trace the successive values of a, b, and c as you "continue" the fraction. Can you discover the patterns that these values follow? Go to the library and research Fibonacci numbers. Write a report on your findings.

46. Explain to a fellow student when you would use the LCM method to add two rational expressions. Give two examples of adding two rational expressions, one in which you use the LCM and the other in which you do not.

47. Which of the two methods given in the text for simplifying complex rational expressions do you prefer? Write a brief paragraph stating the reasons for your choice.

A.6 Solving Equations

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Factoring Polynomials (Appendix A, Section A.3, pp. A28–A29)
- Zero-Product Property (Appendix A, Section A.1, p. A4)
- Square Roots (Appendix A, Section A.1, pp. A9–A10)
- Absolute Value (Appendix A, Section A.1, pp. A5–A6)

Now Work the 'Are You Prepared?' problems on page A51.

OBJECTIVES
1. Solve Equations by Factoring (p. A46)
2. Solve Equations Involving Absolute Value (p. A46)
3. Solve a Quadratic Equation by Factoring (p. A47)
4. Solve a Quadratic Equation by Completing the Square (p. A48)
5. Solve a Quadratic Equation Using the Quadratic Formula (p. A49)

An **equation in one variable** is a statement in which two expressions, at least one containing the variable, are equal. The expressions are called the **sides** of the equation. Since an equation is a statement, it may be true or false, depending on the value of the variable. Unless otherwise restricted, the admissible values of the variable are those in the domain of the variable. The admissible values of the variable, if any, that result in a true statement are called **solutions,** or **roots,** of the equation. To **solve an equation** means to find all the solutions of the equation.

For example, the following are all equations in one variable, x:

$$x + 5 = 9 \qquad x^2 + 5x = 2x - 2 \qquad \frac{x^2 - 4}{x + 1} = 0 \qquad \sqrt{x^2 + 9} = 5$$

The first of these statements, $x + 5 = 9$, is true when $x = 4$ and false for any other choice of x. That is, 4 is a solution of the equation $x + 5 = 9$. We also say that 4 **satisfies** the equation $x + 5 = 9$, because, when we substitute 4 for x, a true statement results.

Sometimes an equation will have more than one solution. For example, the equation

$$\frac{x^2 - 4}{x + 1} = 0$$

has $x = -2$ and $x = 2$ as solutions.

Usually, we will write the solution of an equation in set notation. This set is called the **solution set** of the equation. For example, the solution set of the equation $x^2 - 9 = 0$ is $\{-3, 3\}$.

Some equations have no real solution. For example, $x^2 + 9 = 5$ has no real solution, because there is no real number whose square when added to 9 equals 5.

An equation that is satisfied for every value of the variable for which both sides are defined is called an **identity.** For example, the equation

$$3x + 5 = x + 3 + 2x + 2$$

is an identity, because this statement is true for any real number x.

One method for solving an equation is to replace the original equation by a succession of equivalent equations until an equation with an obvious solution is obtained.

For example, all the following equations are equivalent.

$$2x + 3 = 13$$
$$2x = 10$$
$$x = 5$$

We conclude that the solution set of the original equation is $\{5\}$.

How do we obtain equivalent equations? In general, there are five ways.

Procedures That Result in Equivalent Equations

1. Interchange the two sides of the equation:

   Replace $3 = x$ by $x = 3$

2. Simplify the sides of the equation by combining like terms, eliminating parentheses, and so on:

   Replace $(x + 2) + 6 = 2x + (x + 1)$

   by $x + 8 = 3x + 1$

3. Add or subtract the same expression on both sides of the equation:

   Replace $3x - 5 = 4$

   by $(3x - 5) + 5 = 4 + 5$

4. Multiply or divide both sides of the equation by the same nonzero expression:

   Replace $\dfrac{3x}{x-1} = \dfrac{6}{x-1} \qquad x \neq 1$

   by $\dfrac{3x}{x-1} \cdot (x - 1) = \dfrac{6}{x-1} \cdot (x - 1)$

5. If one side of the equation is 0 and the other side can be factored, then we may use the Zero-Product Property* and set each factor equal to 0:

   Replace $x(x - 3) = 0$

   by $x = 0$ or $x - 3 = 0$

WARNING Squaring both sides of an equation does not necessarily lead to an equivalent equation. ■

Whenever it is possible to solve an equation in your head, do so. For example,

The solution of $2x = 8$ is $x = 4$.

The solution of $3x - 15 = 0$ is $x = 5$.

Now Work PROBLEM 13

Often, though, some rearrangement is necessary.

EXAMPLE 1 **Solving an Equation**

Solve the equation: $3x - 5 = 4$

Solution Replace the original equation by a succession of equivalent equations.

$$3x - 5 = 4$$
$$(3x - 5) + 5 = 4 + 5 \qquad \text{Add 5 to both sides.}$$
$$3x = 9 \qquad \text{Simplify.}$$
$$\frac{3x}{3} = \frac{9}{3} \qquad \text{Divide both sides by 3.}$$
$$x = 3 \qquad \text{Simplify.}$$

The last equation, $x = 3$, has the single solution 3. All these equations are equivalent, so 3 is the only solution of the original equation, $3x - 5 = 4$.

* The Zero-Product Property says that if $ab = 0$, then $a = 0$ or $b = 0$ or both equal 0.

✓**Check:** It is a good practice to check the solution by substituting 3 for x in the original equation.

$$3x - 5 = 3(3) - 5 = 9 - 5 = 4$$

The solution checks.

Now Work PROBLEMS 27 AND 33

1 Solve Equations by Factoring

EXAMPLE 2 **Solving Equations by Factoring**

Solve the equations: (a) $x^3 = 4x$ (b) $x^3 - x^2 - 4x + 4 = 0$

Solution (a) Begin by collecting all terms on one side. This results in 0 on one side and an expression to be factored on the other.

$$x^3 = 4x$$
$$x^3 - 4x = 0$$
$$x(x^2 - 4) = 0 \quad \text{Factor.}$$
$$x(x - 2)(x + 2) = 0 \quad \text{Factor again.}$$
$$x = 0 \text{ or } x - 2 = 0 \text{ or } x + 2 = 0 \quad \text{Apply the Zero-Product Property.}$$
$$x = 0 \text{ or } x = 2 \text{ or } x = -2 \quad \text{Solve for } x.$$

The solution set is $\{-2, 0, 2\}$.

✓**Check:**
$x = -2$: $(-2)^3 = -8$ and $4(-2) = -8$ — -2 is a solution.
$x = 0$: $0^3 = 0$ and $4 \cdot 0 = 0$ — 0 is a solution.
$x = 2$: $2^3 = 8$ and $4 \cdot 2 = 8$ — 4 is a solution.

(b) Group the terms of $x^3 - x^2 - 4x + 4 = 0$ as follows:

$$(x^3 - x^2) - (4x - 4) = 0$$

Factor out x^2 from the first grouping and 4 from the second.

$$x^2(x - 1) - 4(x - 1) = 0$$

This reveals the common factor $(x - 1)$, so we have

$$(x^2 - 4)(x - 1) = 0$$
$$(x - 2)(x + 2)(x - 1) = 0 \quad \text{Factor again.}$$
$$x - 2 = 0 \text{ or } x + 2 = 0 \quad x - 1 = 0 \quad \text{Apply the Zero-Product Property.}$$
$$x = 2 \quad x = -2 \quad x = 1 \quad \text{Solve for } x.$$

The solution set is $\{-2, 1, 2\}$.

✓**Check:**

$x = -2$: $(-2)^3 - (-2)^2 - 4(-2) + 4 = -8 - 4 + 8 + 4 = 0$ — -2 is a solution
$x = 1$: $1^3 - 1^2 - 4(1) + 4 = 1 - 1 - 4 + 4 = 0$ — 1 is a solution.
$x = 2$: $2^3 - 2^2 - 4(2) + 4 = 8 - 4 - 8 + 4 = 0$ — 2 is a solution.

Now Work PROBLEM 37

2 Solve Equations Involving Absolute Value

On the real number line, there are two points whose distance from the origin is 5 units, -5 and 5, so the equation $|x| = 5$ will have the solution set $\{-5, 5\}$.

EXAMPLE 3 **Solving an Equation Involving Absolute Value**

Solve the equation: $|x + 4| = 13$

Solution There are two possibilities.

$$x + 4 = 13 \quad \text{or} \quad x + 4 = -13$$
$$x = 9 \quad \text{or} \quad x = -17$$

The solution set is $\{-17, 9\}$.

Now Work PROBLEM 49

3 Solve a Quadratic Equation by Factoring

DEFINITION

A **quadratic equation** is an equation equivalent to one of the form

$$ax^2 + bx + c = 0 \quad (1)$$

where a, b, and c are real numbers and $a \neq 0$.

A quadratic equation written in the form $ax^2 + bx + c = 0$ is said to be in **standard form.**

Sometimes, a quadratic equation is called a **second-degree equation,** because the left side is a polynomial of degree 2.

When a quadratic equation is written in standard form $ax^2 + bx + c = 0$, it may be possible to factor the expression on the left side into the product of two first-degree polynomials. Then, by using the Zero-Product Property and setting each factor equal to 0, we can solve the resulting linear equations and obtain the solutions of the quadratic equation.

EXAMPLE 4 **Solving a Quadratic Equation by Factoring**

Solve the equation: $2x^2 = x + 3$

Solution Put the equation $2x^2 = x + 3$ in standard form by adding $-x - 3$ to both sides.

$$2x^2 = x + 3$$
$$2x^2 - x - 3 = 0 \quad \text{Add } -x - 3 \text{ to both sides.}$$

The left side may now be factored as

$$(2x - 3)(x + 1) = 0 \quad \text{Factor.}$$

so that

$$2x - 3 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{Apply the Zero-Product Property.}$$
$$x = \frac{3}{2} \qquad x = -1 \quad \text{Solve.}$$

The solution set is $\left\{-1, \frac{3}{2}\right\}$.

When the left side factors into two linear equations with the same solution, the quadratic equation is said to have a **repeated solution.** We also call this solution a **root of multiplicity 2,** or a **double root.**

EXAMPLE 5 **Solving a Quadratic Equation by Factoring**

Solve the equation: $9x^2 - 6x + 1 = 0$

Solution This equation is already in standard form, and the left side can be factored.

$$9x^2 - 6x + 1 = 0$$
$$(3x - 1)(3x - 1) = 0 \quad \text{Factor.}$$

so

$$x = \frac{1}{3} \quad \text{or} \quad x = \frac{1}{3} \quad \text{Solve for } x.$$

This equation has only the repeated solution $\frac{1}{3}$. The solution set is $\left\{\frac{1}{3}\right\}$.

Now Work PROBLEM 67

The Square Root Method

Suppose that we wish to solve the quadratic equation

$$x^2 = p \tag{2}$$

where $p \geq 0$ is a nonnegative number. Proceed as in the earlier examples.

$$x^2 - p = 0 \quad \text{Put in standard form.}$$
$$(x - \sqrt{p})(x + \sqrt{p}) = 0 \quad \text{Factor (over the real numbers).}$$
$$x = \sqrt{p} \quad \text{or} \quad x = -\sqrt{p} \quad \text{Solve.}$$

We have the following result:

> If $x^2 = p$ and $p \geq 0$, then $x = \sqrt{p}$ or $x = -\sqrt{p}$. **(3)**

When statement (3) is used, it is called the **Square Root Method.** In statement (3), note that if $p > 0$ the equation $x^2 = p$ has two solutions, $x = \sqrt{p}$ and $x = -\sqrt{p}$. We usually abbreviate these solutions as $x = \pm\sqrt{p}$, read as "x equals plus or minus the square root of p."

For example, the two solutions of the equation

$$x^2 = 4$$

are

$$x = \pm\sqrt{4} \quad \text{Use the Square Root Method}$$

and, since $\sqrt{4} = 2$, we have

$$x = \pm 2$$

The solution set is $\{-2, 2\}$.

Now Work PROBLEM 81

4 Solve a Quadratic Equation by Completing the Square

EXAMPLE 6 **Solving a Quadratic Equation by Completing the Square**

Solve by completing the square: $2x^2 - 8x - 5 = 0$

Solution First, rewrite the equation as follows:

$$2x^2 - 8x - 5 = 0$$
$$2x^2 - 8x = 5$$

Next, divide both sides by 2 so that the coefficient of x^2 is 1. (This enables us to complete the square at the next step.)

$$x^2 - 4x = \frac{5}{2}$$

Finally, complete the square by adding $\left[\frac{1}{2}(-4)\right]^2 = 4$ to both sides.

$$x^2 - 4x + 4 = \frac{5}{2} + 4$$

$$(x-2)^2 = \frac{13}{2}$$

$$x - 2 = \pm\sqrt{\frac{13}{2}} \quad \text{Use the Square Root Method.}$$

$$x - 2 = \pm\frac{\sqrt{26}}{2} \quad \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}$$

$$x = 2 \pm \frac{\sqrt{26}}{2}$$

COMMENT If we wanted an approximation, say rounded to two decimal places, of these solutions, we would use a calculator to get $\{-0.55, 4.55\}$. ■

The solution set is $\left\{2 - \frac{\sqrt{26}}{2}, 2 + \frac{\sqrt{26}}{2}\right\}$

Now Work PROBLEM 85

5 Solve a Quadratic Equation Using the Quadratic Formula

We can use the method of completing the square to obtain a general formula for solving any quadratic equation

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

COMMENT There is no loss in generality to assume that $a > 0$, since if $a < 0$ we can multiply by -1 to obtain an equivalent equation with a positive leading coefficient. ■

As in Example 6, we rearrange the terms as

$$ax^2 + bx = -c \quad a > 0$$

Since $a > 0$, we can divide both sides by a to get

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Now the coefficient of x^2 is 1. To complete the square on the left side, add the square of $\frac{1}{2}$ of the coefficient of x; that is, add

$$\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$$

to both sides. Then

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2}{4a^2} - \frac{4ac}{4a^2} = \frac{b^2 - 4ac}{4a^2} \qquad \textbf{(4)}$$

Provided that $b^2 - 4ac \geq 0$, we now can use the Square Root Method to get

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$ The square root of a quotient equals the quotient of the square roots. Also, $\sqrt{4a^2} = 2a$ since $a > 0$.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$ Add $-\frac{b}{2a}$ to both sides.

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$ Combine the quotients on the right.

What if $b^2 - 4ac$ is negative? Then equation (4) states that the left expression (a real number squared) equals the right expression (a negative number). Since this occurrence is impossible for real numbers, we conclude that if $b^2 - 4ac < 0$ the quadratic equation has no *real* solution. (We discuss quadratic equations for which the quantity $b^2 - 4ac < 0$ in detail in the next section.)

We now state the *quadratic formula.*

THEOREM

Consider the quadratic equation

$$ax^2 + bx + c = 0 \qquad a \neq 0$$

If $b^2 - 4ac < 0$, this equation has no real solution.

If $b^2 - 4ac \geq 0$, the real solution(s) of this equation is (are) given by the **quadratic formula.**

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (5)$$

The quantity $b^2 - 4ac$ is called the **discriminant** of the quadratic equation, because its value tells us whether the equation has real solutions. In fact, it also tells us how many solutions to expect.

Discriminant of a Quadratic Equation

For a quadratic equation $ax^2 + bx + c = 0$:

1. If $b^2 - 4ac > 0$, there are two unequal real solutions.
2. If $b^2 - 4ac = 0$, there is a repeated solution, a root of multiplicity 2.
3. If $b^2 - 4ac < 0$, there is no real solution.

When asked to find the real solutions, if any, of a quadratic equation, always evaluate the discriminant first to see how many real solutions there are.

EXAMPLE 7 Solving a Quadratic Equation Using the Quadratic Formula

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 - 5x + 1 = 0$$

Solution The equation is in standard form, so we compare it to $ax^2 + bx + c = 0$ to find a, b, and c.

$$3x^2 - 5x + 1 = 0$$
$$ax^2 + bx + c = 0 \qquad a = 3,\ b = -5,\ c = 1$$

With $a = 3, b = -5$, and $c = 1$, evaluate the discriminant $b^2 - 4ac$.

$$b^2 - 4ac = (-5)^2 - 4(3)(1) = 25 - 12 = 13$$

Since $b^2 - 4ac > 0$, there are two real solutions, which can be found using the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-5) \pm \sqrt{13}}{2(3)} = \frac{5 \pm \sqrt{13}}{6}$$

The solution set is $\left\{\frac{5 - \sqrt{13}}{6}, \frac{5 + \sqrt{13}}{6}\right\}$.

EXAMPLE 8 **Solving a Quadratic Equation Using the Quadratic Formula**

Use the quadratic formula to find the real solutions, if any, of the equation

$$3x^2 + 2 = 4x$$

Solution The equation, as given, is not in standard form.

$$3x^2 + 2 = 4x$$
$$3x^2 - 4x + 2 = 0 \quad \text{Put in standard form.}$$
$$ax^2 + bx + c = 0 \quad \text{Compare to standard form.}$$

With $a = 3, b = -4$, and $c = 2$, we find

$$b^2 - 4ac = (-4)^2 - 4(3)(2) = 16 - 24 = -8$$

Since $b^2 - 4ac < 0$, the equation has no real solution.

Now Work PROBLEMS 91 AND 97

SUMMARY Procedure for Solving a Quadratic Equation

To solve a quadratic equation, first put it in standard form:

$$ax^2 + bx + c = 0$$

Then:

STEP 1: Identify a, b, and c.

STEP 2: Evaluate the discriminant, $b^2 - 4ac$.

STEP 3: (a) If the discriminant is negative, the equation has no real solution.
(b) If the discriminant is zero, the equation has one real solution, a repeated root.
(c) If the discriminant is positive, the equation has two distinct real solutions.

If you can easily spot factors, use the factoring method to solve the equation. Otherwise, use the quadratic formula or the method of completing the square.

A.6 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Factor $x^2 - 5x - 6$. (pp. A28–A29)

2. Factor $2x^2 - x - 3$. (pp. A28–A29)

3. The solution set of the equation $(x - 3)(3x + 5) = 0$ is ________. (p. A4)

4. ***True or False*** $\sqrt{x^2} = |x|$. (pp. A9–A10)

Concepts and Vocabulary

5. *True or False* Squaring both sides of an equation results in an equivalent equation.
6. An equation that is satisfied for every choice of the variable for which both sides are defined is called a(n) ________.
7. *True or False* The solution of the equation $3x - 8 = 0$ is $\frac{3}{8}$.
8. *True or False* Some equations have no solution.
9. To solve the equation $x^2 + 5x = 0$ by completing the square, you would _____ the number _____ to both sides.
10. The quantity $b^2 - 4ac$ is called the ________ of a quadratic equation. If it is ________, the equation has no real solution.
11. *True or False* Quadratic equations always have two real solutions.
12. *True or False* If the discriminant of a quadratic equation is positive, then the equation has two solutions that are negatives of one another.

Skill Building

In Problems 13–78, solve each equation.

13. $3x = 21$
14. $3x = -24$
15. $5x + 15 = 0$
16. $3x + 18 = 0$
17. $2x - 3 = 5$
18. $3x + 4 = -8$
19. $\frac{1}{3}x = \frac{5}{12}$
20. $\frac{2}{3}x = \frac{9}{2}$
21. $6 - x = 2x + 9$
22. $3 - 2x = 2 - x$
23. $2(3 + 2x) = 3(x - 4)$
24. $3(2 - x) = 2x - 1$
25. $8x - (2x + 1) = 3x - 10$
26. $5 - (2x - 1) = 10$
27. $\frac{1}{2}x - 4 = \frac{3}{4}x$
28. $1 - \frac{1}{2}x = 5$
29. $0.9t = 0.4 + 0.1t$
30. $0.9t = 1 + t$
31. $\frac{2}{y} + \frac{4}{y} = 3$
32. $\frac{4}{y} - 5 = \frac{5}{2y}$
33. $(x + 7)(x - 1) = (x + 1)^2$
34. $(x + 2)(x - 3) = (x - 3)^2$
35. $z(z^2 + 1) = 3 + z^3$
36. $w(4 - w^2) = 8 - w^3$
37. $x^2 = 9x$
38. $x^3 = x^2$
39. $t^3 - 9t^2 = 0$
40. $4z^3 - 8z^2 = 0$
41. $\frac{3}{2x - 3} = \frac{2}{x + 5}$
42. $\frac{-2}{x + 4} = \frac{-3}{x + 1}$
43. $(x + 2)(3x) = (x + 2)(6)$
44. $(x - 5)(2x) = (x - 5)(4)$
45. $\frac{2}{x - 2} = \frac{3}{x + 5} + \frac{10}{(x + 5)(x - 2)}$
46. $\frac{1}{2x + 3} + \frac{1}{x - 1} = \frac{1}{(2x + 3)(x - 1)}$
47. $|2x| = 6$
48. $|3x| = 12$
49. $|2x + 3| = 5$
50. $|3x - 1| = 2$
51. $|1 - 4t| = 5$
52. $|1 - 2z| = 3$
53. $|-2x| = 8$
54. $|-x| = 1$
55. $|-2|x = 4$
56. $|3|x = 9$
57. $|x - 2| = -\frac{1}{2}$
58. $|2 - x| = -1$
59. $|x^2 - 4| = 0$
60. $|x^2 - 9| = 0$
61. $|x^2 - 2x| = 3$
62. $|x^2 + x| = 12$
63. $|x^2 + x - 1| = 1$
64. $|x^2 + 3x - 2| = 2$
65. $x^2 = 4x$
66. $x^2 = -8x$
67. $z^2 + 4z - 12 = 0$
68. $v^2 + 7v + 12 = 0$
69. $2x^2 - 5x - 3 = 0$
70. $3x^2 + 5x + 2 = 0$
71. $x(x - 7) + 12 = 0$
72. $x(x + 1) = 12$
73. $4x^2 + 9 = 12x$
74. $25x^2 + 16 = 40x$
75. $6x - 5 = \frac{6}{x}$
76. $x + \frac{12}{x} = 7$
77. $\frac{4(x - 2)}{x - 3} + \frac{3}{x} = \frac{-3}{x(x - 3)}$
78. $\frac{5}{x + 4} = 4 + \frac{3}{x - 2}$

In Problems 79–84, solve each equation by the Square Root Method.

79. $x^2 = 25$
80. $x^2 = 36$
81. $(x - 1)^2 = 4$
82. $(x + 2)^2 = 1$
83. $(2y + 3)^2 = 9$
84. $(3x - 2)^2 = 4$

In Problems 85–90, solve each equation by completing the square.

85. $x^2 + 4x = 21$
86. $x^2 - 6x = 13$
87. $x^2 - \frac{1}{2}x - \frac{3}{16} = 0$
88. $x^2 + \frac{2}{3}x - \frac{1}{3} = 0$
89. $3x^2 + x - \frac{1}{2} = 0$
90. $2x^2 - 3x - 1 = 0$

In Problems 91–102, find the real solutions, if any, of each equation. Use the quadratic formula.

91. $x^2 - 4x + 2 = 0$ **92.** $x^2 + 4x + 2 = 0$ **93.** $x^2 - 5x - 1 = 0$

94. $x^2 + 5x + 3 = 0$ **95.** $2x^2 - 5x + 3 = 0$ **96.** $2x^2 + 5x + 3 = 0$

97. $4y^2 - y + 2 = 0$ **98.** $4t^2 + t + 1 = 0$ **99.** $4x^2 = 1 - 2x$

100. $2x^2 = 1 - 2x$ **101.** $x^2 + \sqrt{3}x - 3 = 0$ **102.** $x^2 + \sqrt{2}x - 2 = 0$

In Problems 103–108, use the discriminant to determine whether each quadratic equation has two unequal real solutions, a repeated real solution, or no real solution without solving the equation.

103. $x^2 - 5x + 7 = 0$ **104.** $x^2 + 5x + 7 = 0$ **105.** $9x^2 - 30x + 25 = 0$

106. $25x^2 - 20x + 4 = 0$ **107.** $3x^2 + 5x - 8 = 0$ **108.** $2x^2 - 3x - 4 = 0$

Applications and Extensions

In Problems 109–114, solve each equation. The letters a, b, and c are constants.

109. $ax - b = c, \quad a \neq 0$ **110.** $1 - ax = b, \quad a \neq 0$ **111.** $\frac{x}{a} + \frac{x}{b} = c, \quad a \neq 0, b \neq 0, a \neq -b$

112. $\frac{a}{x} + \frac{b}{x} = c, \quad c \neq 0$ **113.** $\frac{1}{x-a} + \frac{1}{x+a} = \frac{2}{x-1}$ **114.** $\frac{b+c}{x+a} = \frac{b-c}{x-a}, \quad c \neq 0, a \neq 0$

Problems 115–120 list some formulas that occur in applications. Solve each formula for the indicated variable.

115. Electricity $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$ for R

116. Finance $A = P(1 + rt)$ for r

117. Mechanics $F = \frac{mv^2}{R}$ for R

118. Chemistry $PV = nRT$ for T

119. Mathematics $S = \frac{a}{1-r}$ for r

120. Mechanics $v = -gt + v_0$ for t

121. Show that the sum of the roots of a quadratic equation is $-\frac{b}{a}$.

122. Show that the product of the roots of a quadratic equation is $\frac{c}{a}$.

123. Find k such that the equation $kx^2 + x + k = 0$ has a repeated real solution.

124. Find k such that the equation $x^2 - kx + 4 = 0$ has a repeated real solution.

125. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the negatives of the real solutions of the equation $ax^2 - bx + c = 0$. Assume that $b^2 - 4ac \geq 0$.

126. Show that the real solutions of the equation $ax^2 + bx + c = 0$ are the reciprocals of the real solutions of the equation $cx^2 + bx + a = 0$. Assume that $b^2 - 4ac \geq 0$.

Explaining Concepts: Discussion and Writing

127. Which of the following pairs of equations are equivalent? Explain.

(a) $x^2 = 9; \quad x = 3$

(b) $x = \sqrt{9}; \quad x = 3$

(c) $(x-1)(x-2) = (x-1)^2; \quad x - 2 = x - 1$

128. The equation

$$\frac{5}{x+3} + 3 = \frac{8+x}{x+3}$$

has no solution, yet when we go through the process of solving it we obtain $x = -3$. Write a brief paragraph to explain what causes this to happen.

129. Make up an equation that has no solution and give it to a fellow student to solve. Ask the fellow student to write a critique of your equation.

130. Describe three ways you might solve a quadratic equation. State your preferred method; explain why you chose it.

131. Explain the benefits of evaluating the discriminant of a quadratic equation before attempting to solve it.

132. Make up three quadratic equations: one having two distinct solutions, one having no real solution, and one having exactly one real solution.

133. The word *quadratic* seems to imply four (*quad*), yet a quadratic equation is an equation that involves a polynomial of degree 2. Investigate the origin of the term *quadratic* as it is used in the expression *quadratic equation*. Write a brief essay on your findings.

'Are You Prepared?' Answers

1. $(x-6)(x+1)$ **2.** $(2x-3)(x+1)$ **3.** $\left\{-\frac{5}{3}, 3\right\}$ **4.** True

A.7 Complex Numbers; Quadratic Equations in the Complex Number System

OBJECTIVES
1 Add, Subtract, Multiply, and Divide Complex Numbers (p. A54)
2 Solve Quadratic Equations in the Complex Number System (p. A58)

Complex Numbers

One property of a real number is that its square is nonnegative. For example, there is no real number x for which

$$x^2 = -1$$

To remedy this situation, we introduce a new number called the *imaginary unit.*

DEFINITION

The **imaginary unit,** which we denote by i, is the number whose square is -1. That is,

$$i^2 = -1$$

This should not surprise you. If our universe were to consist only of integers, there would be no number x for which $2x = 1$. This unfortunate circumstance was remedied by introducing numbers such as $\frac{1}{2}$ and $\frac{2}{3}$, the *rational numbers*. If our universe were to consist only of rational numbers, there would be no x whose square equals 2. That is, there would be no number x for which $x^2 = 2$. To remedy this, we introduced numbers such as $\sqrt{2}$ and $\sqrt[3]{5}$, the *irrational numbers*. The *real numbers,* you will recall, consist of the rational numbers and the irrational numbers. Now, if our universe were to consist only of real numbers, then there would be no number x whose square is -1. To remedy this, we introduce a number i, whose square is -1.

In the progression outlined, each time we encountered a situation that was unsuitable, we introduced a new number system to remedy this situation. And each new number system contained the earlier number system as a subset. The number system that results from introducing the number i is called the **complex number system.**

DEFINITION

Complex numbers are numbers of the form $a + bi$, where a and b are real numbers. The real number a is called the **real part** of the number $a + bi$; the real number b is called the **imaginary part** of $a + bi$; and i is the imaginary unit, so $i^2 = -1$.

For example, the complex number $-5 + 6i$ has the real part -5 and the imaginary part 6.

When a complex number is written in the form $a + bi$, where a and b are real numbers, we say it is in **standard form.** However, if the imaginary part of a complex number is negative, such as in the complex number $3 + (-2)i$, we agree to write it instead in the form $3 - 2i$.

Also, the complex number $a + 0i$ is usually written merely as a. This serves to remind us that the real numbers are a subset of the complex numbers. The complex number $0 + bi$ is usually written as bi. Sometimes the complex number bi is called a **pure imaginary number.**

1 Add, Subtract, Multiply, and Divide Complex Numbers

Equality, addition, subtraction, and multiplication of complex numbers are defined so as to preserve the familiar rules of algebra for real numbers. Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. That is,

Equality of Complex Numbers

$$a + bi = c + di \qquad \text{if and only if } a = c \text{ and } b = d \tag{1}$$

Two complex numbers are added by forming the complex number whose real part is the sum of the real parts and whose imaginary part is the sum of the imaginary parts. That is,

Sum of Complex Numbers

$$(a + bi) + (c + di) = (a + c) + (b + d)i \tag{2}$$

To subtract two complex numbers, we use this rule:

Difference of Complex Numbers

$$(a + bi) - (c + di) = (a - c) + (b - d)i \tag{3}$$

EXAMPLE 1 **Adding and Subtracting Complex Numbers**

(a) $(3 + 5i) + (-2 + 3i) = [3 + (-2)] + (5 + 3)i = 1 + 8i$

(b) $(6 + 4i) - (3 + 6i) = (6 - 3) + (4 - 6)i = 3 + (-2)i = 3 - 2i$

Now Work PROBLEM 13

Products of complex numbers are calculated as illustrated in Example 2.

EXAMPLE 2 **Multiplying Complex Numbers**

$$(5 + 3i)\cdot(2 + 7i) = 5\cdot(2 + 7i) + 3i(2 + 7i) = 10 + 35i + 6i + 21i^2$$

↑ Distributive Property ↑ Distributive Property

$$= 10 + 41i + 21(-1)$$

↑ $i^2 = -1$

$$= -11 + 41i$$

Based on the procedure of Example 2, we define the **product** of two complex numbers as follows:

Product of Complex Numbers

$$(a + bi)\cdot(c + di) = (ac - bd) + (ad + bc)i \tag{4}$$

Do not bother to memorize formula (4). Instead, whenever it is necessary to multiply two complex numbers, follow the usual rules for multiplying two binomials, as in Example 2, remembering that $i^2 = -1$. For example,

$$(2i)(2i) = 4i^2 = -4$$

$$(2 + i)(1 - i) = 2 - 2i + i - i^2 = 3 - i$$

Now Work PROBLEM 19

Algebraic properties for addition and multiplication, such as the commutative, associative, and distributive properties, hold for complex numbers. The property that every nonzero complex number has a multiplicative inverse, or reciprocal, requires a closer look.

DEFINITION

If $z = a + bi$ is a complex number, then its **conjugate,** denoted by $\bar{z}$, is defined as

$$\bar{z} = \overline{a + bi} = a - bi$$

For example, $\overline{2 + 3i} = 2 - 3i$ and $\overline{-6 - 2i} = -6 + 2i$.

EXAMPLE 3 Multiplying a Complex Number by Its Conjugate

Find the product of the complex number $z = 3 + 4i$ and its conjugate $\bar{z}$.

Solution Since $\bar{z} = 3 - 4i$, we have

$$z\bar{z} = (3 + 4i)(3 - 4i) = 9 - 12i + 12i - 16i^2 = 9 + 16 = 25$$

The result obtained in Example 3 has an important generalization.

THEOREM

The product of a complex number and its conjugate is a nonnegative real number. That is, if $z = a + bi$, then

$$z\bar{z} = a^2 + b^2 \qquad (5)$$

Proof If $z = a + bi$, then

$$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2$$ ■

To express the reciprocal of a nonzero complex number z in standard form, multiply the numerator and denominator of $\frac{1}{z}$ by $\bar{z}$. That is, if $z = a + bi$ is a nonzero complex number, then

$$\frac{1}{a + bi} = \frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{a - bi}{a^2 + b^2}$$

↑ Use (5).

$$= \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2}i$$

EXAMPLE 4 Writing the Reciprocal of a Complex Number in Standard Form

Write $\frac{1}{3 + 4i}$ in standard form $a + bi$; that is, find the reciprocal of $3 + 4i$.

Solution The idea is to multiply the numerator and denominator by the conjugate of $3 + 4i$, that is, by the complex number $3 - 4i$. The result is

$$\frac{1}{3 + 4i} = \frac{1}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{3 - 4i}{9 + 16} = \frac{3}{25} - \frac{4}{25}i$$

To express the quotient of two complex numbers in standard form, multiply the numerator and denominator of the quotient by the conjugate of the denominator.

EXAMPLE 5 **Writing the Quotient of Two Complex Numbers in Standard Form**

Write each of the following in standard form.

(a) $\dfrac{1+4i}{5-12i}$ (b) $\dfrac{2-3i}{4-3i}$

Solution (a) $\dfrac{1+4i}{5-12i} = \dfrac{1+4i}{5-12i}\cdot\dfrac{5+12i}{5+12i} = \dfrac{5+12i+20i+48i^2}{25+144}$

$= \dfrac{-43+32i}{169} = -\dfrac{43}{169} + \dfrac{32}{169}i$

(b) $\dfrac{2-3i}{4-3i} = \dfrac{2-3i}{4-3i}\cdot\dfrac{4+3i}{4+3i} = \dfrac{8+6i-12i-9i^2}{16+9}$

$= \dfrac{17-6i}{25} = \dfrac{17}{25} - \dfrac{6}{25}i$

Now Work PROBLEM 27

EXAMPLE 6 **Writing Other Expressions in Standard Form**

If $z = 2 - 3i$ and $w = 5 + 2i$, write each of the following expressions in standard form.

(a) $\dfrac{z}{w}$ (b) $\overline{z+w}$ (c) $z + \overline{z}$

Solution (a) $\dfrac{z}{w} = \dfrac{z\cdot\overline{w}}{w\cdot\overline{w}} = \dfrac{(2-3i)(5-2i)}{(5+2i)(5-2i)} = \dfrac{10-4i-15i+6i^2}{25+4}$

$= \dfrac{4-19i}{29} = \dfrac{4}{29} - \dfrac{19}{29}i$

(b) $\overline{z+w} = \overline{(2-3i)+(5+2i)} = \overline{7-i} = 7+i$

(c) $z + \overline{z} = (2-3i) + (2+3i) = 4$

The conjugate of a complex number has certain general properties that we shall find useful later.

For a real number $a = a + 0i$, the conjugate is $\overline{a} = \overline{a+0i} = a - 0i = a$. That is,

THEOREM The conjugate of a real number is the real number itself.

Other properties of the conjugate that are direct consequences of the definition are given next. In each statement, z and w represent complex numbers.

THEOREM The conjugate of the conjugate of a complex number is the complex number itself.

$$(\overline{\overline{z}}) = z \tag{6}$$

The conjugate of the sum of two complex numbers equals the sum of their conjugates.

$$\overline{z+w} = \overline{z} + \overline{w} \tag{7}$$

The conjugate of the product of two complex numbers equals the product of their conjugates.

$$\overline{z\cdot w} = \overline{z}\cdot\overline{w} \tag{8}$$

We leave the proofs of equations (6), (7), and (8) as exercises.

Powers of *i*

The **powers of *i*** follow a pattern that is useful to know.

$$i^1 = i \qquad i^5 = i^4 \cdot i = 1 \cdot i = i$$
$$i^2 = -1 \qquad i^6 = i^4 \cdot i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i \qquad i^7 = i^4 \cdot i^3 = -i$$
$$i^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \qquad i^8 = i^4 \cdot i^4 = 1$$

And so on. The powers of i repeat with every fourth power.

EXAMPLE 7 **Evaluating Powers of *i***

(a) $i^{27} = i^{24} \cdot i^3 = (i^4)^6 \cdot i^3 = 1^6 \cdot i^3 = -i$

(b) $i^{101} = i^{100} \cdot i^1 = (i^4)^{25} \cdot i = 1^{25} \cdot i = i$

EXAMPLE 8 **Writing the Power of a Complex Number in Standard Form**

Write $(2 + i)^3$ in standard form.

Solution Use the special product formula for $(x + a)^3$.

$$(x + a)^3 = x^3 + 3ax^2 + 3a^2x + a^3$$

COMMENT Another way to find $(2 + i)^3$ is to multiply out $(2 + i)^2(2 + i)$. ■

Using this special product formula,

$$\begin{aligned}(2 + i)^3 &= 2^3 + 3 \cdot i \cdot 2^2 + 3 \cdot i^2 \cdot 2 + i^3 \\ &= 8 + 12i + 6(-1) + (-i) \\ &= 2 + 11i.\end{aligned}$$

Now Work PROBLEM 41

2 Solve Quadratic Equations in the Complex Number System

Quadratic equations with a negative discriminant have no real number solution. However, if we extend our number system to allow complex numbers, quadratic equations will always have a solution. Since the solution to a quadratic equation involves the square root of the discriminant, we begin with a discussion of square roots of negative numbers.

DEFINITION If N is a positive real number, we define the **principal square root of $-N$**, denoted by $\sqrt{-N}$, as

$$\sqrt{-N} = \sqrt{N}\,i$$

where i is the imaginary unit and $i^2 = -1$.

WARNING In writing $\sqrt{-N} = \sqrt{N}\,i$ be sure to place i outside the $\sqrt{\ }$ symbol. ■

EXAMPLE 9 **Evaluating the Square Root of a Negative Number**

(a) $\sqrt{-1} = \sqrt{1}\,i = i$ (b) $\sqrt{-4} = \sqrt{4}\,i = 2i$

(c) $\sqrt{-8} = \sqrt{8}\,i = 2\sqrt{2}\,i$

EXAMPLE 10 **Solving Equations**

Solve each equation in the complex number system.

(a) $x^2 = 4$ (b) $x^2 = -9$

Solution (a) $x^2 = 4$

$$x = \pm\sqrt{4} = \pm 2$$

The equation has two solutions, -2 and 2. The solution set is $\{-2, 2\}$.

(b) $x^2 = -9$

$$x = \pm\sqrt{-9} = \pm\sqrt{9}\,i = \pm 3i$$

The equation has two solutions, $-3i$ and $3i$. The solution set is $\{-3i, 3i\}$.

Now Work PROBLEMS 49 AND 53

WARNING When working with square roots of negative numbers, do not set the square root of a product equal to the product of the square roots (which can be done with positive numbers). To see why, look at this calculation: We know that $\sqrt{100} = 10$. However, it is also true that $100 = (-25)(-4)$, so

$$10 = \sqrt{100} = \sqrt{(-25)(-4)} \neq \sqrt{-25}\,\sqrt{-4} = \left(\sqrt{25}\,i\right)\left(\sqrt{4}\,i\right) = (5i)(2i) = 10i^2 = -10$$

↑ Here is the error.

Because we have defined the square root of a negative number, we can now restate the quadratic formula without restriction.

THEOREM **Quadratic Formula**

In the complex number system, the solutions of the quadratic equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$, are given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \textbf{(9)}$$

EXAMPLE 11 **Solving Quadratic Equations in the Complex Number System**

Solve the equation $x^2 - 4x + 8 = 0$ in the complex number system.

Solution Here $a = 1$, $b = -4$, $c = 8$, and $b^2 - 4ac = 16 - 4(1)(8) = -16$. Using equation (9), we find that

$$x = \frac{-(-4) \pm \sqrt{-16}}{2(1)} = \frac{4 \pm \sqrt{16}\,i}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i$$

The equation has two solutions, $2 - 2i$ and $2 + 2i$. The solution set is $\{2 - 2i, 2 + 2i\}$.

✓**Check:** $2 + 2i$: $(2 + 2i)^2 - 4(2 + 2i) + 8 = 4 + 8i + 4i^2 - 8 - 8i + 8$
$= 4 - 4 = 0$

$2 - 2i$: $(2 - 2i)^2 - 4(2 - 2i) + 8 = 4 - 8i + 4i^2 - 8 + 8i + 8$
$= 4 - 4 = 0$

Now Work PROBLEM 59

The discriminant $b^2 - 4ac$ of a quadratic equation still serves as a way to determine the character of the solutions.

Character of the Solutions of a Quadratic Equation

In the complex number system, consider a quadratic equation $ax^2 + bx + c = 0$ with real coefficients.

1. If $b^2 - 4ac > 0$, the equation has two unequal real solutions.
2. If $b^2 - 4ac = 0$, the equation has a repeated real solution, a double root.
3. If $b^2 - 4ac < 0$, the equation has two complex solutions that are not real. The solutions are conjugates of each other.

The third conclusion in the display is a consequence of the fact that if $b^2 - 4ac = -N < 0$ then, by the quadratic formula, the solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{-N}}{2a} = \frac{-b + \sqrt{N}i}{2a} = \frac{-b}{2a} + \frac{\sqrt{N}}{2a}i$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{-N}}{2a} = \frac{-b - \sqrt{N}i}{2a} = \frac{-b}{2a} - \frac{\sqrt{N}}{2a}i$$

which are conjugates of each other.

EXAMPLE 12 **Determining the Character of the Solution of a Quadratic Equation**

Without solving, determine the character of the solution of each equation.

(a) $3x^2 + 4x + 5 = 0$ (b) $2x^2 + 4x + 1 = 0$ (c) $9x^2 - 6x + 1 = 0$

Solution

(a) Here $a = 3$, $b = 4$, and $c = 5$, so $b^2 - 4ac = 16 - 4(3)(5) = -44$. The solutions are two complex numbers that are not real and are conjugates of each other.

(b) Here $a = 2$, $b = 4$, and $c = 1$, so $b^2 - 4ac = 16 - 8 = 8$. The solutions are two unequal real numbers.

(c) Here $a = 9$, $b = -6$, and $c = 1$, so $b^2 - 4ac = 36 - 4(9)(1) = 0$. The solution is a repeated real number, that is, a double root.

Now Work PROBLEM 73

A.7 Assess Your Understanding

Concepts and Vocabulary

1. ***True or False*** The square of a complex number is sometimes negative.
2. $(2 + i)(2 - i) =$ ________.
3. ***True or False*** In the complex number system, a quadratic equation has four solutions.
4. In the complex number $5 + 2i$, the number 5 is called the ________ part; the number 2 is called the ________ part; the number i is called the ________ ________.
5. The equation $x^2 = -4$ has the solution set ________.
6. ***True or False*** The conjugate of $2 + 5i$ is $-2 - 5i$.
7. ***True or False*** All real numbers are complex numbers.
8. ***True or False*** If $2 - 3i$ is a solution of a quadratic equation with real coefficients, then $-2 + 3i$ is also a solution.

Skill Building

In Problems 9–46, write each expression in the standard form $a + bi$.

9. $(2 - 3i) + (6 + 8i)$ **10.** $(4 + 5i) + (-8 + 2i)$ **11.** $(-3 + 2i) - (4 - 4i)$ **12.** $(3 - 4i) - (-3 - 4i)$

13. $(2 - 5i) - (8 + 6i)$ **14.** $(-8 + 4i) - (2 - 2i)$ **15.** $3(2 - 6i)$ **16.** $-4(2 + 8i)$

17. $2i(2 - 3i)$ **18.** $3i(-3 + 4i)$ **19.** $(3 - 4i)(2 + i)$ **20.** $(5 + 3i)(2 - i)$

21. $(-6+i)(-6-i)$ **22.** $(-3+i)(3+i)$ **23.** $\dfrac{10}{3-4i}$ **24.** $\dfrac{13}{5-12i}$

25. $\dfrac{2+i}{i}$ **26.** $\dfrac{2-i}{-2i}$ **27.** $\dfrac{6-i}{1+i}$ **28.** $\dfrac{2+3i}{1-i}$

29. $\left(\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}i\right)^2$ **30.** $\left(\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}i\right)^2$ **31.** $(1+i)^2$ **32.** $(1-i)^2$

33. i^{23} **34.** i^{14} **35.** i^{-15} **36.** i^{-23}

37. i^6-5 **38.** $4+i^3$ **39.** $6i^3-4i^5$ **40.** $4i^3-2i^2+1$

41. $(1+i)^3$ **42.** $(3i)^4+1$ **43.** $i^7(1+i^2)$ **44.** $2i^4(1+i^2)$

45. $i^6+i^4+i^2+1$ **46.** $i^7+i^5+i^3+i$

In Problems 47–52, perform the indicated operations and express your answer in the form $a+bi$.

47. $\sqrt{-4}$ **48.** $\sqrt{-9}$ **49.** $\sqrt{-25}$

50. $\sqrt{-64}$ **51.** $\sqrt{(3+4i)(4i-3)}$ **52.** $\sqrt{(4+3i)(3i-4)}$

In Problems 53–72, solve each equation in the complex number system.

53. $x^2+4=0$ **54.** $x^2-4=0$ **55.** $x^2-16=0$ **56.** $x^2+25=0$

57. $x^2-6x+13=0$ **58.** $x^2+4x+8=0$ **59.** $x^2-6x+10=0$ **60.** $x^2-2x+5=0$

61. $8x^2-4x+1=0$ **62.** $10x^2+6x+1=0$ **63.** $5x^2+1=2x$ **64.** $13x^2+1=6x$

65. $x^2+x+1=0$ **66.** $x^2-x+1=0$ **67.** $x^3-8=0$ **68.** $x^3+27=0$

69. $x^4=16$ **70.** $x^4=1$ **71.** $x^4+13x^2+36=0$ **72.** $x^4+3x^2-4=0$

In Problems 73–78, without solving, determine the character of the solutions of each equation in the complex number system.

73. $3x^2-3x+4=0$ **74.** $2x^2-4x+1=0$ **75.** $2x^2+3x=4$

76. $x^2+6=2x$ **77.** $9x^2-12x+4=0$ **78.** $4x^2+12x+9=0$

79. $2+3i$ is a solution of a quadratic equation with real coefficients. Find the other solution.

80. $4-i$ is a solution of a quadratic equation with real coefficients. Find the other solution.

In Problems 81–84, $z=3-4i$ and $w=8+3i$. Write each expression in the standard form $a+bi$.

81. $z+\bar{z}$ **82.** $w-\bar{w}$ **83.** $z\bar{z}$ **84.** $\overline{z-w}$

Applications and Extensions

85. Electrical Circuits The impedance Z, in ohms, of a circuit element is defined as the ratio of the phasor voltage V, in volts, across the element to the phasor current I, in amperes, through the elements. That is, $Z=\dfrac{V}{I}$. If the voltage across a circuit element is $18+i$ volts and the current through the element is $3-4i$ amperes, determine the impedance.

86. Parallel Circuits In an ac circuit with two parallel pathways, the total impedance Z, in ohms, satisfies the formula $\dfrac{1}{Z}=\dfrac{1}{Z_1}+\dfrac{1}{Z_2}$, where Z_1 is the impedance of the first pathway and Z_2 is the impedance of the second pathway. Determine the total impedance if the impedances of the two pathways are $Z_1=2+i$ ohms and $Z_2=4-3i$ ohms.

87. Use $z=a+bi$ to show that $z+\bar{z}=2a$ and $z-\bar{z}=2bi$.

88. Use $z=a+bi$ to show that $\bar{\bar{z}}=z$.

89. Use $z=a+bi$ and $w=c+di$ to show that $\overline{z+w}=\bar{z}+\bar{w}$.

90. Use $z=a+bi$ and $w=c+di$ to show that $\overline{z\cdot w}=\bar{z}\cdot\bar{w}$.

Explaining Concepts: Discussion and Writing

91. Explain to a friend how you would add two complex numbers and how you would multiply two complex numbers. Explain any differences in the two explanations.

92. Write a brief paragraph that compares the method used to rationalize the denominator of a radical expression and the method used to write the quotient of two complex numbers in standard form.

93. Use an Internet search engine to investigate the origins of complex numbers. Write a paragraph describing what you find and present it to the class.

94. What Went Wrong? A student multiplied $\sqrt{-9}$ and $\sqrt{-9}$ as follows:

$$\sqrt{-9}\cdot\sqrt{-9}=\sqrt{(-9)(-9)}=\sqrt{81}=9$$

The instructor marked the problem incorrect. Why?

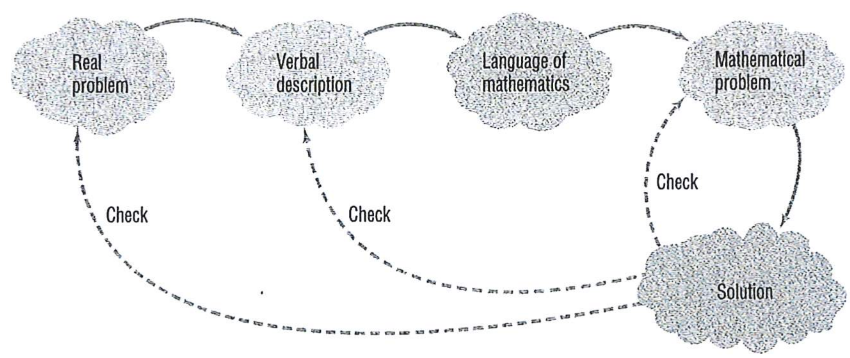
A.8 Problem Solving: Interest, Mixture, Uniform Motion, Constant Rate Job Applications

The icon is a Model It! icon. It indicates that the discussion or problem involves modeling.

Applied (word) problems do not come in the form "Solve the equation. . . . " Instead, they supply information using words, a verbal description of the real problem. So, to solve applied problems, we must be able to translate the verbal description into the language of mathematics. We do this by using variables to represent unknown quantities and then finding relationships (such as equations) that involve these variables. The process of doing all this is called **mathematical modeling.**

Any solution to the mathematical problem must be checked against the mathematical problem, the verbal description, and the real problem. See Figure 23 for an illustration of the **modeling process.**

Figure 23



1 Translate Verbal Descriptions into Mathematical Expressions

Let's look at a few examples that will help you to translate certain words into mathematical symbols.

EXAMPLE 1 Translating Verbal Descriptions into Mathematical Expressions

(a) The (average) speed of an object equals the distance traveled divided by the time required.

Translation: If r is the speed, d the distance, and t the time, then $r = \frac{d}{t}$.

(b) Let x denote a number.

The number 5 times as large as x is $5x$.

The number 3 less than x is $x - 3$.

The number that exceeds x by 4 is $x + 4$.

The number that, when added to x, gives 5 is $5 - x$.

Now Work PROBLEM 7

Always check the units used to measure the variables of an applied problem. In Example 1(a), if v is measured in miles per hour, then the distance s must be expressed in miles and the time t must be expressed in hours. It is a good practice to check units to be sure that they are consistent and make sense.

Steps for Solving Applied Problems

STEP 1: Read the problem carefully, perhaps two or three times. Pay particular attention to the question being asked in order to identify what you are looking for. If you can, determine realistic possibilities for the answer.

STEP 2: Assign a letter (variable) to represent what you are looking for, and, if necessary, express any remaining unknown quantities in terms of this variable.

STEP 3: Make a list of all the known facts, and translate them into mathematical expressions. These may take the form of an equation or an inequality involving the variable. If possible, draw an appropriately labeled diagram to assist you. Sometimes a table or chart helps.

STEP 4: Solve the equation for the variable, and then answer the question.

STEP 5: Check the answer with the facts in the problem. If it agrees, congratulations! If it does not agree, try again.

2 Solve Interest Problems

Interest is money paid for the use of money. The total amount borrowed (whether by an individual from a bank in the form of a loan or by a bank from an individual in the form of a savings account) is called the **principal.** The **rate of interest,** expressed as a percent, is the amount charged for the use of the principal for a given period of time, usually on a yearly (that is, on a per annum) basis.

Simple Interest Formula

If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is

$$I = Prt \tag{1}$$

Interest charged according to formula (1) is called **simple interest.** When using formula (1), be sure to express r as a decimal. For example, if the rate of interest is 4%, then $r = 0.04$.

EXAMPLE 2 **Finance: Computing Interest on a Loan**

Suppose that Juanita borrows \$500 for 6 months at the simple interest rate of 9% per annum. What is the interest that Juanita will be charged on the loan? How much does Juanita owe after 6 months?

Solution The rate of interest is given per annum, so the actual time that the money is borrowed must be expressed in years. The interest charged would be the principal, \$500, times the rate of interest (9% = 0.09) times the time in years, $\frac{1}{2}$:

$$\text{Interest charged} = I = Prt = (500)(0.09)\left(\frac{1}{2}\right) = \$22.50$$

After 6 months, Juanita will owe what she borrowed plus the interest:

$$\$500 + \$22.50 = \$522.50$$

EXAMPLE 3

Financial Planning

Candy has \$70,000 to invest and wants an annual return of \$2800, which requires an overall rate of return of 4%. She can invest in a safe, government-insured certificate of deposit, but it only pays 2%. To obtain 4%, she agrees to invest some of her money in noninsured corporate bonds paying 7%. How much should be placed in each investment to achieve her goal?

Solution

Step 1: The question is asking for two dollar amounts: the principal to invest in the corporate bonds and the principal to invest in the certificate of deposit.

Step 2: Let x represent the amount (in dollars) to be invested in the bonds. Then $70{,}000 - x$ is the amount that will be invested in the certificate. (Do you see why?)

Step 3: Now set up Table 1:

Table 1

	Principal (\$)	Rate	Time (yr)	Interest (\$)
Bonds	x	$7\% = 0.07$	1	$0.07x$
Certificate	$70{,}000 - x$	$2\% = 0.02$	1	$0.02(70{,}000 - x)$
Total	70,000	$4\% = 0.04$	1	$0.04(70{,}000) = 2800$

Since the total interest from the investments is equal to $0.04(70{,}000) = 2800$, we have the equation

$$0.07x + 0.02(70{,}000 - x) = 2800$$

(Note that the units are consistent: the unit is dollars on each side.)

Step 4: $0.07x + 1400 - 0.02x = 2800$

$$0.05x = 1400 \quad \text{Simplify.}$$

$$x = 28{,}000 \quad \text{Divide both sides by 0.05.}$$

Candy should place \$28,000 in the bonds and \$70,000 − \$28,000 = \$42,000 in the certificate.

Step 5: The interest on the bonds after 1 year is 0.07(\$28,000) = \$1960; the interest on the certificate after 1 year is 0.02(\$42,000) = \$840. The total annual interest is \$2800, the required amount.

Now Work PROBLEM 17

3 Solve Mixture Problems

Oil refineries sometimes produce gasoline that is a blend of two or more types of fuel; bakeries occasionally blend two or more types of flour for their bread. These problems are referred to as **mixture problems** because they combine two or more quantities to form a mixture.

EXAMPLE 4

Blending Coffees

The manager of a local coffee shop decides to experiment with a new blend of coffee. She will mix some B grade Colombian coffee that sells for \$5 per pound with some A grade Arabica coffee that sells for \$10 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be \$7 per pound, and there is to be no difference in revenue from selling the new blend versus selling the other types. How many pounds of the B grade Colombian and A grade Arabica coffees are required?

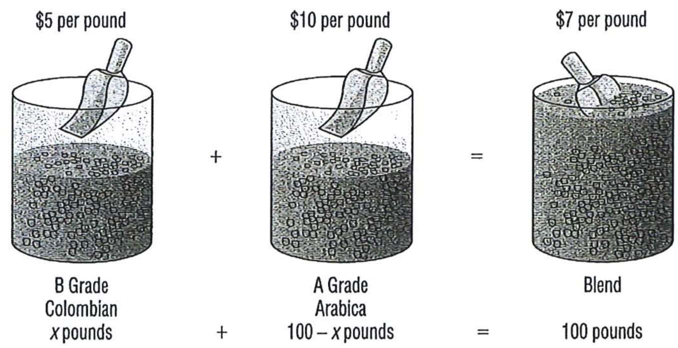
Solution

Step 1: The question is asking how many pounds of Colombian coffee and how many pounds of Arabica coffee are needed to make 100 pounds of the mixture.

Step 2: Let x represent the number of pounds of the B grade Colombian coffee. Then $100 - x$ equals the number of pounds of the A grade Arabica coffee.

STEP 3: See Figure 24.

Figure 24



Since there is to be no difference in revenue between selling the A and B grades separately versus the blend, we have

$$\begin{Bmatrix}\text{Price per pound}\\ \text{of B grade}\end{Bmatrix}\begin{Bmatrix}\text{\# Pounds}\\ \text{B grade}\end{Bmatrix} + \begin{Bmatrix}\text{Price per pound}\\ \text{of A grade}\end{Bmatrix}\begin{Bmatrix}\text{\# Pounds}\\ \text{A grade}\end{Bmatrix} = \begin{Bmatrix}\text{Price per pound}\\ \text{of blend}\end{Bmatrix}\begin{Bmatrix}\text{\# Pounds}\\ \text{blend}\end{Bmatrix}$$

$$\$5 \cdot x + \$10 \cdot (100 - x) = \$7 \cdot 100$$

STEP 4: Solve the equation

$$\begin{aligned} 5x + 10(100 - x) &= 700 \\ 5x + 1000 - 10x &= 700 \\ -5x &= -300 \\ x &= 60 \end{aligned}$$

The manager should blend 60 pounds of B grade Colombian coffee with $100 - 60 = 40$ pounds of A grade Arabica coffee to get the desired blend.

STEP 5: ✓**Check:** The 60 pounds of B grade coffee would sell for $(\$5)(60) = \$300$, and the 40 pounds of A grade coffee would sell for $(\$10)(40) = \$400$; the total revenue, \$700, equals the revenue obtained from selling the blend, as desired.

Now Work PROBLEM 21

4 Solve Uniform Motion Problems

Objects that move at a constant speed are said to be in **uniform motion.** When the average speed of an object is known, it can be interpreted as its constant speed. For example, a bicyclist traveling at an average speed of 25 miles per hour can be modeled as uniform motion with a constant speed of 25 miles per hour.

Uniform Motion Formula

If an object moves at an average speed (rate) r, the distance d covered in time t is given by the formula

$$d = rt \qquad \textbf{(2)}$$

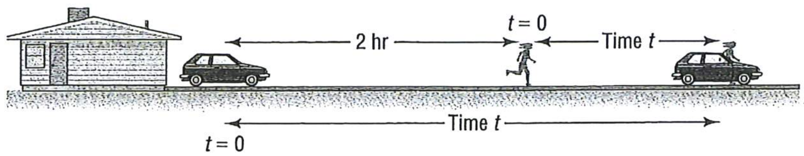
That is, Distance = Rate · Time.

EXAMPLE 5 Physics: Uniform Motion

Tanya, who is a long-distance runner, runs at an average speed of 8 miles per hour (mi/hr). Two hours after Tanya leaves your house, you leave in your Honda and follow the same route. If your average speed is 40 mi/hr, how long will it be before you catch up to Tanya? How far will each of you be from your home?

Solution Refer to Figure 25. Use t to represent the time (in hours) that it takes the Honda to catch up to Tanya. When this occurs, the total time elapsed for Tanya is $t + 2$ hours.

Figure 25



Set up Table 2:

Table 2

	Rate (mi/hr)	Time (hr)	Distance (mi)
Tanya	8	$t+2$	$8(t+2)$
Honda	40	t	$40t$

Since the distance traveled is the same, we are led to the following equation:

$$\begin{aligned} 8(t+2) &= 40t \\ 8t + 16 &= 40t \\ 32t &= 16 \\ t &= \frac{1}{2} \text{ hour} \end{aligned}$$

It will take the Honda $\frac{1}{2}$ hour to catch up to Tanya. Each will have gone 20 miles.

Check: In 2.5 hours, Tanya travels a distance of $(2.5)(8) = 20$ miles. In $\frac{1}{2}$ hour, the Honda travels a distance of $\left(\frac{1}{2}\right)(40) = 20$ miles.

EXAMPLE 6 Physics: Uniform Motion

A motorboat heads upstream a distance of 24 miles on a river whose current is running at 3 miles per hour (mi/hr). The trip up and back takes 6 hours. Assuming that the motorboat maintained a constant speed relative to the water, what was its speed?

Solution See Figure 26. We use r to represent the constant speed of the motorboat relative to the water. Then the true speed going upstream is $r - 3$ mi/hr, and the true speed going downstream is $r + 3$ mi/hr. Since Distance = Rate × Time, then $\text{Time} = \frac{\text{Distance}}{\text{Rate}}$. Set up Table 3.

Figure 26

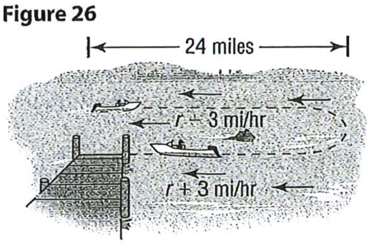


Table 3

	Rate (mi/hr)	Distance (mi)	Time = $\frac{\text{Distance}}{\text{Rate}}$ (hr)
Upstream	$r-3$	24	$\frac{24}{r-3}$
Downstream	$r+3$	24	$\frac{24}{r+3}$

Since the total time up and back is 6 hours, we have

$$\frac{24}{r-3} + \frac{24}{r+3} = 6$$

$$\frac{24(r+3) + 24(r-3)}{(r-3)(r+3)} = 6 \quad \text{Add the quotients on the left.}$$

$$\frac{48r}{r^2 - 9} = 6 \quad \text{Simplify.}$$

$$48r = 6(r^2 - 9) \quad \text{Multiply both sides by } r^2 - 9.$$

$$6r^2 - 48r - 54 = 0 \quad \text{Place in standard form.}$$

$$r^2 - 8r - 9 = 0 \quad \text{Divide by 6.}$$

$$(r-9)(r+1) = 0 \quad \text{Factor.}$$

$$r = 9 \quad \text{or} \quad r = -1 \quad \text{Apply the Zero-Product Property and solve.}$$

Discard the solution $r = -1$ mi/hr, so the speed of the motorboat relative to the water is 9 mi/hr.

Now Work PROBLEM 27

5 Solve Constant Rate Job Problems

This section involves jobs that are performed at a **constant rate.** Our assumption is that, if a job can be done in t units of time, then $\frac{1}{t}$ of the job is done in 1 unit of time.

EXAMPLE 7 Working Together to Do a Job

At 10 AM Danny is asked by his father to weed the garden. From past experience, Danny knows that this will take him 4 hours, working alone. His older brother, Mike, when it is his turn to do this job, requires 6 hours. Since Mike wants to go golfing with Danny and has a reservation for 1 PM, he agrees to help Danny. Assuming no gain or loss of efficiency, when will they finish if they work together? Can they make the golf date?

Solution We set up Table 4. In 1 hour, Danny does $\frac{1}{4}$ of the job, and in 1 hour, Mike does $\frac{1}{6}$ of the job. Let t be the time (in hours) that it takes them to do the job together. In 1 hour, then, $\frac{1}{t}$ of the job is completed. We reason as follows:

Table 4

	Hours to Do Job	Part of Job Done in 1 Hour
Danny	4	$\frac{1}{4}$
Mike	6	$\frac{1}{6}$
Together	t	$\frac{1}{t}$

$$\left(\begin{matrix}\text{Part done by Danny}\\ \text{in 1 hour}\end{matrix}\right) + \left(\begin{matrix}\text{Part done by Mike}\\ \text{in 1 hour}\end{matrix}\right) = \left(\begin{matrix}\text{Part done together}\\ \text{in 1 hour}\end{matrix}\right)$$

From Table 4,

$$\frac{1}{4} + \frac{1}{6} = \frac{1}{t}$$

$$\frac{3}{12} + \frac{2}{12} = \frac{1}{t}$$

$$\frac{5}{12} = \frac{1}{t}$$

$$5t = 12$$

$$t = \frac{12}{5}$$

Working together, the job can be done in $\frac{12}{5}$ hours, or 2 hours, 24 minutes. They should make the golf date, since they will finish at 12:24 PM.

Now Work PROBLEM 33

The next example is one that you will probably see again in a slightly different form if you study calculus.

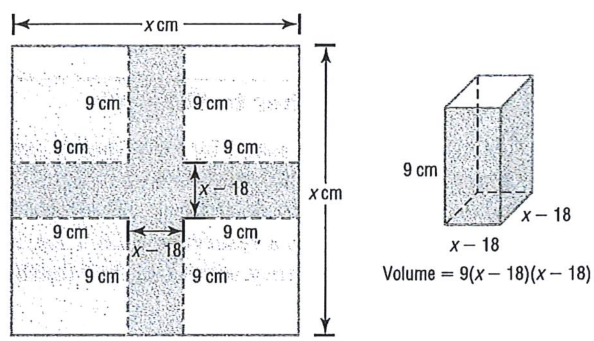
EXAMPLE 8 **Constructing a Box**

From each corner of a square piece of sheet metal, remove a square of side 9 centimeters. Turn up the edges to form an open box. If the box is to hold 144 cubic centimeters (cm^3), what should be the dimensions of the piece of sheet metal?

Solution We use Figure 27 as a guide. We have labeled by x the length of a side of the square piece of sheet metal. The box will be of height 9 centimeters, and its square base will measure $x - 18$ on each side. The volume V (Length $\times$ Width $\times$ Height) of the box is therefore

$$V = (x - 18)(x - 18) \cdot 9 = 9(x - 18)^2$$

Figure 27



Since the volume of the box is to be 144 cm^3, we have

$$\begin{aligned} 9(x - 18)^2 &= 144 && V = 144 \\ (x - 18)^2 &= 16 && \text{Divide each side by 9.} \\ x - 18 &= \pm 4 && \text{Use the Square Root Method.} \\ x &= 18 \pm 4 \\ x = 22 \quad &\text{or} \quad x = 14 \end{aligned}$$

We discard the solution $x = 14$ (do you see why?) and conclude that the sheet metal should be 22 centimeters by 22 centimeters.

✓ **Check:** If we begin with a piece of sheet metal 22 centimeters by 22 centimeters, cut out a 9-centimeter square from each corner, and fold up the edges, we get a box whose dimensions are 9 by 4 by 4, with volume $9 \times 4 \times 4 = 144\ \text{cm}^3$, as required.

Now Work PROBLEM 55

A.8 Assess Your Understanding

Concepts and Vocabulary

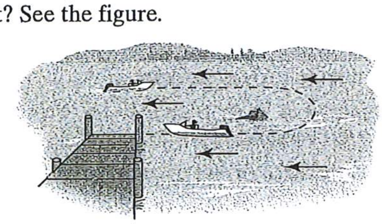
1. The process of using variables to represent unknown quantities and then finding relationships that involve these variables is referred to as ____________ __________.
2. The money paid for the use of money is ________.
3. Objects that move at a constant rate are said to be in ________ ________.
4. ***True or False*** The amount charged for the use of principal for a given period of time is called the rate of interest.
5. ***True or False*** If an object moves at an average speed r, the distance d covered in time t is given by the formula $d = rt$.
6. Suppose that you want to mix two coffees in order to obtain 100 pounds of the blend. If x represents the number of pounds of coffee A, write an algebraic expression that represents the number of pounds of coffee B.

Applications and Extensions

In Problems 7–16, translate each sentence into a mathematical equation. Be sure to identify the meaning of all symbols.

7. **Geometry** The area of a circle is the product of the number π and the square of the radius.
8. **Geometry** The circumference of a circle is the product of the number π and twice the radius.
9. **Geometry** The area of a square is the square of the length of a side.
10. **Geometry** The perimeter of a square is four times the length of a side.
11. **Physics** Force equals the product of mass and acceleration.
12. **Physics** Pressure is force per unit area.
13. **Physics** Work equals force times distance.
14. **Physics** Kinetic energy is one-half the product of the mass and the square of the velocity.
15. **Business** The total variable cost of manufacturing x dishwashers is \$150 per dishwasher times the number of dishwashers manufactured.
16. **Business** The total revenue derived from selling x dishwashers is \$250 per dishwasher times the number of dishwashers sold.
17. **Financial Planning** Betsy, a recent retiree, requires \$6000 per year in extra income. She has \$50,000 to invest and can invest in B-rated bonds paying 15% per year or in a certificate of deposit (CD) paying 7% per year. How much money should be invested in each to realize exactly \$6000 in interest per year?
18. **Financial Planning** After 2 years, Betsy (see Problem 17) finds that she will now require \$7000 per year. Assuming that the remaining information is the same, how should the money be reinvested?
19. **Banking** A bank loaned out \$12,000, part of it at the rate of 8% per year and the rest at the rate of 18% per year. If the interest received totaled \$1000, how much was loaned at 8%?
20. **Banking** Wendy, a loan officer at a bank, has \$1,000,000 to lend and is required to obtain an average return of 18% per year. If she can lend at the rate of 19% or at the rate of 16%, how much can she lend at the 16% rate and still meet her requirement?
21. **Blending Teas** The manager of a store that specializes in selling tea decides to experiment with a new blend. She will mix some Earl Grey tea that sells for \$5 per pound with some Orange Pekoe tea that sells for \$3 per pound to get 100 pounds of the new blend. The selling price of the new blend is to be \$4.50 per pound, and there is to be no difference in revenue from selling the new blend versus selling the other types. How many pounds of the Earl Grey tea and Orange Pekoe tea are required?
22. **Business: Blending Coffee** A coffee manufacturer wants to market a new blend of coffee that sells for \$3.90 per pound by mixing two coffees that sell for \$2.75 and \$5 per pound, respectively. What amounts of each coffee should be blended to obtain the desired mixture?

    [**Hint:** Assume that the total weight of the desired blend is 100 pounds.]
23. **Business: Mixing Nuts** A nut store normally sells cashews for \$9.00 per pound and almonds for \$3.50 per pound. But at the end of the month the almonds had not sold well, so, in order to sell 60 pounds of almonds, the manager decided to mix the 60 pounds of almonds with some cashews and sell the mixture for \$7.50 per pound. How many pounds of cashews should be mixed with the almonds to ensure no change in the profit?
24. **Business: Mixing Candy** A candy store sells boxes of candy containing caramels and cremes. Each box sells for \$12.50 and holds 30 pieces of candy (all pieces are the same size). If the caramels cost \$0.25 to produce and the cremes cost \$0.45 to produce, how many of each should be in a box to make a profit of \$3?
25. **Physics: Uniform Motion** A motorboat can maintain a constant speed of 16 miles per hour relative to the water. The boat makes a trip upstream to a certain point in 20 minutes; the return trip takes 15 minutes. What is the speed of the current? See the figure.



26. **Physics: Uniform Motion** A motorboat heads upstream on a river that has a current of 3 miles per hour. The trip upstream takes 5 hours, and the return trip takes 2.5 hours. What is the speed of the motorboat? (Assume that the motorboat maintains a constant speed relative to the water.)
27. **Physics: Uniform Motion** A motorboat maintained a constant speed of 15 miles per hour relative to the water in going 10 miles upstream and then returning. The total time for

the trip was 1.5 hours. Use this information to find the speed of the current.

28. Physics: Uniform Motion Two cars enter the Florida Turnpike at Commercial Boulevard at 8:00 AM, each heading for Wildwood. One car's average speed is 10 miles per hour more than the other's. The faster car arrives at Wildwood at 11:00 AM, $\frac{1}{2}$ hour before the other car. What was the average speed of each car? How far did each travel?

29. Moving Walkways The speed of a moving walkway is typically about 2.5 feet per second. Walking on such a moving walkway, it takes Karen a total of 40 seconds to travel 50 feet with the movement of the walkway and then back again against the movement of the walkway. What is Karen's normal walking speed?

Source: Answers.com

30. Moving Walkways The Gare Montparnasse train station in Paris has a high-speed version of a moving walkway. If he walks while riding this moving walkway, Jean Claude can travel 200 meters in 30 seconds less time than if he stands still on the moving walkway. If Jean Claude walks at a normal rate of 1.5 meters per second, what is the speed of the Gare Montparnasse walkway?

Source: Answers.com

31. Tennis A regulation doubles tennis court has an area of 2808 square feet. If it is 6 feet longer than twice its width, determine the dimensions of the court.

Source: United States Tennis Association

32. Laser Printers It takes an HP LaserJet 1300 laser printer 10 minutes longer to complete a 600-page print job by itself than it takes an HP LaserJet 2420 to complete the same job by itself. Together the two printers can complete the job in 12 minutes. How long does it take each printer to complete the print job alone? What is the speed of each printer?

Source: Hewlett-Packard

33. Working Together on a Job Trent can deliver his newspapers in 30 minutes. It takes Lois 20 minutes to do the same route. How long would it take them to deliver the newspapers if they work together?

34. Working Together on a Job Patrick, by himself, can paint four rooms in 10 hours. If he hires April to help, they can do the same job together in 6 hours. If he lets April work alone, how long will it take her to paint four rooms?

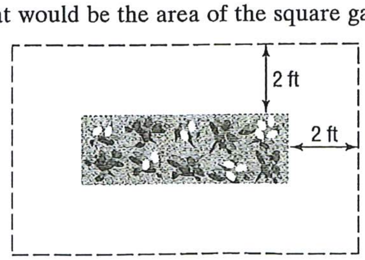
35. Enclosing a Garden A gardener has 46 feet of fencing to be used to enclose a rectangular garden that has a border 2 feet wide surrounding it. See the figure.

(a) If the length of the garden is to be twice its width, what will be the dimensions of the garden?

(b) What is the area of the garden?

(c) If the length and width of the garden are to be the same, what would be the dimensions of the garden?

(d) What would be the area of the square garden?



36. Construction A pond is enclosed by a wooden deck that is 3 feet wide. The fence surrounding the deck is 100 feet long.

(a) If the pond is square, what are its dimensions?

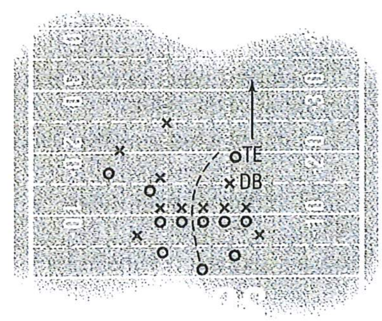
(b) If the pond is rectangular and the length of the pond is to be three times its width, what are its dimensions?

(c) If the pond is circular, what is its diameter?

(d) Which pond has the most area?

37. Football A tight end can run the 100-yard dash in 12 seconds. A defensive back can do it in 10 seconds. The tight end catches a pass at his own 20-yard line with the defensive back at the 15-yard line. (See the figure.) If no other players are nearby, at what yard line will the defensive back catch up to the tight end?

[**Hint:** At time $t = 0$, the defensive back is 5 yards behind the tight end.]



38. Computing Business Expense Therese, an outside salesperson, uses her car for both business and pleasure. Last year, she traveled 30,000 miles, using 900 gallons of gasoline. Her car gets 40 miles per gallon on the highway and 25 in the city. She can deduct all highway travel, but no city travel, on her taxes. How many miles should Therese be allowed as a business expense?

39. Mixing Water and Antifreeze How much water should be added to 1 gallon of pure antifreeze to obtain a solution that is 60% antifreeze?

40. Mixing Water and Antifreeze The cooling system of a certain foreign-made car has a capacity of 15 liters. If the system is filled with a mixture that is 40% antifreeze, how much of this mixture should be drained and replaced by pure antifreeze so that the system is filled with a solution that is 60% antifreeze?

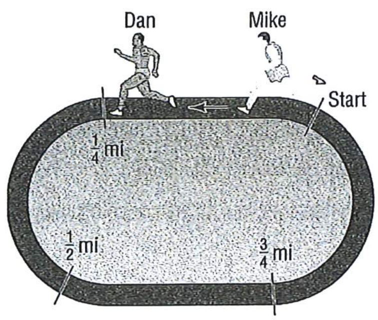
41. Chemistry: Salt Solutions How much water must be evaporated from 32 ounces of a 4% salt solution to make a 6% salt solution?

42. Chemistry: Salt Solutions How much water must be evaporated from 240 gallons of a 3% salt solution to produce a 5% salt solution?

43. Purity of Gold The purity of gold is measured in karats, with pure gold being 24 karats. Other purities of gold are expressed as proportional parts of pure gold. Thus, 18-karat gold is $\frac{18}{24}$, or 75% pure gold; 12-karat gold is $\frac{12}{24}$, or 50% pure gold; and so on. How much 12-karat gold should be mixed with pure gold to obtain 60 grams of 16-karat gold?

44. Chemistry: Sugar Molecules A sugar molecule has twice as many atoms of hydrogen as it does oxygen and one more atom of carbon than oxygen. If a sugar molecule has a total of 45 atoms, how many are oxygen? How many are hydrogen?

45. Running a Race Mike can run the mile in 6 minutes, and Dan can run the mile in 9 minutes. If Mike gives Dan a head start of 1 minute, how far from the start will Mike pass Dan? How long does it take? See the figure.



46. Range of an Airplane An air rescue plane averages 300 miles per hour in still air. It carries enough fuel for 5 hours of flying time. If, upon takeoff, it encounters a head wind of 30 mi/hr, how far can it fly and return safely? (Assume that the wind speed remains constant.)

47. Emptying Oil Tankers An oil tanker can be emptied by the main pump in 4 hours. An auxiliary pump can empty the tanker in 9 hours. If the main pump is started at 9 AM, when is the latest the auxiliary pump can be started so that the tanker is emptied by noon?

48. Cement Mix A 20-pound bag of Economy brand cement mix contains 25% cement and 75% sand. How much pure cement must be added to produce a cement mix that is 40% cement?

49. Emptying a Tub A bathroom tub will fill in 15 minutes with both faucets open and the stopper in place. With both faucets closed and the stopper removed, the tub will empty in 20 minutes. How long will it take for the tub to fill if both faucets are open and the stopper is removed?

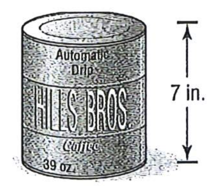
50. Using Two Pumps A 5-horsepower (hp) pump can empty a pool in 5 hours. A smaller, 2-hp pump empties the same pool in 8 hours. The pumps are used together to begin emptying this pool. After two hours, the 2-hp pump breaks down. How long will it take the larger pump to empty the pool?

51. A Biathlon Suppose that you have entered an 87-mile biathlon that consists of a run and a bicycle race. During your run, your average speed is 6 miles per hour, and during your bicycle race, your average speed is 25 miles per hour. You finish the race in 5 hours. What is the distance of the run? What is the distance of the bicycle race?

52. Cyclists Two cyclists leave a city at the same time, one going east and the other going west. The westbound cyclist bikes 5 mph faster than the eastbound cyclist. After 6 hours they are 246 miles apart. How fast is each cyclist riding?

53. Comparing Olympic Heroes In the 1984 Olympics, C. Lewis of the United States won the gold medal in the 100-meter race with a time of 9.99 seconds. In the 1896 Olympics, Thomas Burke, also of the United States, won the gold medal in the 100-meter race in 12.0 seconds. If they ran in the same race repeating their respective times, by how many meters would Lewis beat Burke?

54. Constructing a Coffee Can A 39-ounce can of Hills Bros.® coffee requires 188.5 square inches of aluminum. If its height is 7 inches, what is its radius? [**Hint:** The surface area S of a right cylinder is $S = 2\pi r^2 + 2\pi rh$, where r is the radius and h is the height.]



55. Constructing a Box An open box is to be constructed from a square piece of sheet metal by removing a square of side 1 foot from each corner and turning up the edges. If the box is to hold 4 cubic feet, what should be the dimension of the sheet metal?

56. Constructing a Box Rework Problem 55 if the piece of sheet metal is a rectangle whose length is twice its width.

Explaining Concepts: Discussion and Writing

57. Critical Thinking You are the manager of a clothing store and have just purchased 100 dress shirts for $20.00 each. After 1 month of selling the shirts at the regular price, you plan to have a sale giving 40% off the original selling price. However, you still want to make a profit of $4 on each shirt at the sale price. What should you price the shirts at initially to ensure this? If, instead of 40% off at the sale, you give 50% off, by how much is your profit reduced?

58. Critical Thinking Make up a word problem that requires solving a linear equation as part of its solution. Exchange problems with a friend. Write a critique of your friend's problem.

59. Critical Thinking Without solving, explain what is wrong with the following mixture problem: How many liters of 25% ethanol should be added to 20 liters of 48% ethanol to obtain a solution of 58% ethanol? Now go through an algebraic solution. What happens?

60. Computing Average Speed In going from Chicago to Atlanta, a car averages 45 miles per hour, and in going from Atlanta to Miami, it averages 55 miles per hour. If Atlanta is halfway between Chicago and Miami, what is the average speed from Chicago to Miami? Discuss an intuitive solution. Write a paragraph defending your intuitive solution. Then solve the problem algebraically. Is your intuitive solution the same as the algebraic one? If not, find the flaw.

61. Speed of a Plane On a recent flight from Phoenix to Kansas City, a distance of 919 nautical miles, the plane arrived 20 minutes early. On leaving the aircraft, I asked the captain, "What was our tail wind?" He replied, "I don't know, but our ground speed was 550 knots." How can you determine if enough information is provided to find the tail wind? If possible, find the tail wind. (1 knot = 1 nautical mile per hour)

A.9 Interval Notation; Solving Inequalities

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Algebra Essentials (Appendix A, Section A.1, pp. A1–A10)

Now Work the 'Are You Prepared?' problems on page A78.

Suppose that a and b are two real numbers and $a < b$. We use the notation $a < x < b$ to mean that x is a number *between* a and b. The expression $a < x < b$ is equivalent to the two inequalities $a < x$ and $x < b$. Similarly, the expression $a \leq x \leq b$ is equivalent to the two inequalities $a \leq x$ and $x \leq b$. The remaining two possibilities, $a \leq x < b$ and $a < x \leq b$, are defined similarly.

Although it is acceptable to write $3 \geq x \geq 2$, it is preferable to reverse the inequality symbols and write instead $2 \leq x \leq 3$ so that, as you read from left to right, the values go from smaller to larger.

A statement such as $2 \leq x \leq 1$ is false because there is no number x for which $2 \leq x$ and $x \leq 1$. Finally, we never mix inequality symbols, as in $2 \leq x \geq 3$.

1 Use Interval Notation

DEFINITION

Let a and b represent two real numbers with $a < b$.

A **closed interval,** denoted by **$[a, b]$,** consists of all real numbers x for which $a \leq x \leq b$.

An **open interval,** denoted by **(a, b),** consists of all real numbers x for which $a < x < b$.

The **half-open,** or **half-closed, intervals** are **$(a, b]$,** consisting of all real numbers x for which $a < x \leq b$, and **$[a, b)$,** consisting of all real numbers x for which $a \leq x < b$.

In each of these definitions, a is called the **left endpoint** and b the **right endpoint** of the interval.

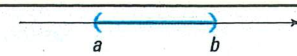
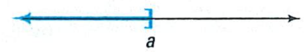
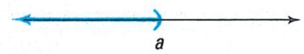
The symbol ∞ (read as "infinity") is not a real number, but a notational device used to indicate unboundedness in the positive direction. The symbol $-\infty$ (read as "negative infinity") also is not a real number, but a notational device used to indicate unboundedness in the negative direction. Using the symbols ∞ and $-\infty$, we can define five other kinds of intervals:

$[a, \infty)$	Consists of all real numbers x for which $x \geq a$
(a, ∞)	Consists of all real numbers x for which $x > a$
$(-\infty, a]$	Consists of all real numbers x for which $x \leq a$
$(-\infty, a)$	Consists of all real numbers x for which $x < a$
$(-\infty, \infty)$	Consists of all real numbers x

Note that ∞ and $-\infty$ are never included as endpoints, since neither is a real number.

Table 5 summarizes interval notation, corresponding inequality notation, and their graphs.

Table 5

Interval	Inequality	Graph
The open interval (a, b)	$a < x < b$	
The closed interval $[a, b]$	$a \le x \le b$	
The half-open interval $[a, b)$	$a \le x < b$	
The half-open interval $(a, b]$	$a < x \le b$	
The interval $[a, \infty)$	$x \ge a$	
The interval (a, ∞)	$x > a$	
The interval $(-\infty, a]$	$x \le a$	
The interval $(-\infty, a)$	$x < a$	
The interval $(-\infty, \infty)$	All real numbers	

EXAMPLE 1 Writing Inequalities Using Interval Notation

Write each inequality using interval notation.

(a) $1 \le x \le 3$ (b) $-4 < x < 0$ (c) $x > 5$ (d) $x \le 1$

Solution

(a) $1 \le x \le 3$ describes all numbers x between 1 and 3, inclusive. In interval notation, we write $[1, 3]$.

(b) In interval notation, $-4 < x < 0$ is written $(-4, 0)$.

(c) $x > 5$ consists of all numbers x greater than 5. In interval notation, we write $(5, \infty)$.

(d) In interval notation, $x \le 1$ is written $(-\infty, 1]$.

EXAMPLE 2 Writing Intervals Using Inequality Notation

Write each interval as an inequality involving x.

(a) $[1, 4)$ (b) $(2, \infty)$ (c) $[2, 3]$ (d) $(-\infty, -3]$

Solution

(a) $[1, 4)$ consists of all numbers x for which $1 \le x < 4$.

(b) $(2, \infty)$ consists of all numbers x for which $x > 2$.

(c) $[2, 3]$ consists of all numbers x for which $2 \le x \le 3$.

(d) $(-\infty, -3]$ consists of all numbers x for which $x \le -3$.

Now Work PROBLEMS 11, 23, AND 31

2 Use Properties of Inequalities

The product of two positive real numbers is positive, the product of two negative real numbers is positive, and the product of 0 and 0 is 0. For any real number a, the value of a^2 is 0 or positive; that is, a^2 is nonnegative. This is called the **nonnegative property**.

In Words
The square of a real number is never negative.

Nonnegative Property

For any real number a,

$$a^2 \ge 0 \qquad (1)$$

If we add the same number to both sides of an inequality, we obtain an equivalent inequality. For example, since $3 < 5$, then $3 + 4 < 5 + 4$ or $7 < 9$. This is called the **addition property** of inequalities.

Addition Property of Inequalities

For real numbers a, b, and c,

If $a < b$, then $a + c < b + c$.	**(2a)**
If $a > b$, then $a + c > b + c$.	**(2b)**

The addition property states that the sense, or direction, of an inequality remains unchanged if the same number is added to each side.

Now let's see what happens if we multiply each side of an inequality by a nonzero number. We begin with $3 < 7$ and multiply each side by 2. The numbers 6 and 14 that result obey the inequality $6 < 14$.

Now start with $9 > 2$ and multiply each side by -4. The numbers -36 and -8 that result obey the inequality $-36 < -8$.

Note that the effect of multiplying both sides of $9 > 2$ by the negative number -4 is that the direction of the inequality symbol is reversed. We are led to the following general **multiplication properties** for inequalities:

Multiplication Properties for Inequalities

For real numbers a, b, and c,

If $a < b$ and if $c > 0$, then $ac < bc$. If $a < b$ and if $c < 0$, then $ac > bc$.	**(3a)**
If $a > b$ and if $c > 0$, then $ac > bc$. If $a > b$ and if $c < 0$, then $ac < bc$.	**(3b)**

In Words
Multiplying by a negative number reverses the inequality.

The multiplication properties state that the sense, or direction, of an inequality *remains the same* if each side is multiplied by a *positive* real number, whereas the direction is *reversed* if each side is multiplied by a *negative* real number.

EXAMPLE 3 Multiplication Property of Inequalities

(a) If $2x < 6$, then $\frac{1}{2}(2x) < \frac{1}{2}(6)$ or $x < 3$.

(b) If $\frac{x}{-3} > 12$, then $-3\left(\frac{x}{-3}\right) < -3(12)$ or $x < -36$.

(c) If $-4x < -8$, then $\frac{-4x}{-4} > \frac{-8}{-4}$ or $x > 2$.

(d) If $-x > 8$, then $(-1)(-x) < (-1)(8)$ or $x < -8$.

Now Work PROBLEM 45

The **reciprocal property** states that the reciprocal of a positive real number is positive and that the reciprocal of a negative real number is negative.

In Words
The reciprocal property states that the reciprocal of a positive real number is positive and that the reciprocal of a negative real number is negative.

Reciprocal Property for Inequalities

If $a > 0$, then $\frac{1}{a} > 0$	If $\frac{1}{a} > 0$, then $a > 0$	**(4a)**
If $a < 0$, then $\frac{1}{a} < 0$	If $\frac{1}{a} < 0$, then $a < 0$	**(4b)**

3 Solve Inequalities

An **inequality in one variable** is a statement involving two expressions, at least one containing the variable, separated by one of the inequality symbols $<$, $\leq$, $>$, or $\geq$. To **solve an inequality** means to find all values of the variable for which the statement is true. These values are called **solutions** of the inequality.

For example, the following are all inequalities involving one variable x:

$$x + 5 < 8 \qquad 2x - 3 \geq 4 \qquad x^2 - 1 \leq 3 \qquad \frac{x+1}{x-2} > 0$$

As with equations, one method for solving an inequality is to replace it by a series of equivalent inequalities until an inequality with an obvious solution, such as $x < 3$, is obtained. We obtain equivalent inequalities by applying some of the same properties as those used to find equivalent equations. The addition property and the multiplication properties form the basis for the following procedures.

Procedures That Leave the Inequality Symbol Unchanged

1. Simplify both sides of the inequality by combining like terms and eliminating parentheses:

   Replace $(x + 2) + 6 > 2x + 5(x + 1)$

   by $x + 8 > 7x + 5$

2. Add or subtract the same expression on both sides of the inequality:

   Replace $3x - 5 < 4$

   by $(3x - 5) + 5 < 4 + 5$

3. Multiply or divide both sides of the inequality by the same positive expression:

   Replace $4x > 16$ by $\dfrac{4x}{4} > \dfrac{16}{4}$

Procedures That Reverse the Sense or Direction of the Inequality Symbol

1. Interchange the two sides of the inequality:

   Replace $3 < x$ by $x > 3$

2. Multiply or divide both sides of the inequality by the same *negative* expression:

   Replace $-2x > 6$ by $\dfrac{-2x}{-2} < \dfrac{6}{-2}$

As the examples that follow illustrate, we solve inequalities using many of the same steps that we would use to solve equations. In writing the solution of an inequality, we may use either set notation or interval notation, whichever is more convenient.

EXAMPLE 4 **Solving an Inequality**

Solve the inequality: $4x + 7 \geq 2x - 3$

Graph the solution set.

Solution

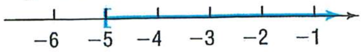
$$4x + 7 \geq 2x - 3$$

$$4x + 7 - 7 \geq 2x - 3 - 7 \quad \text{Subtract 7 from both sides.}$$

$$4x \geq 2x - 10 \quad \text{Simplify.}$$

$$4x - 2x \geq 2x - 10 - 2x \quad \text{Subtract } 2x \text{ from both sides.}$$
$$2x \geq -10 \quad \text{Simplify.}$$
$$\frac{2x}{2} \geq \frac{-10}{2} \quad \text{Divide both sides by 2. (The direction of the inequality symbol is unchanged.)}$$
$$x \geq -5 \quad \text{Simplify.}$$

Figure 28



The solution set is $\{x|x \geq -5\}$ or, using interval notation, all numbers in the interval $[-5, \infty)$. See Figure 28 for the graph.

Now Work PROBLEM 57

4 Solve Combined Inequalities

EXAMPLE 5 Solving Combined Inequalities

Solve the inequality: $-5 < 3x - 2 < 1$
Graph the solution set.

Solution Recall that the inequality

$$-5 < 3x - 2 < 1$$

is equivalent to the two inequalities

$$-5 < 3x - 2 \quad \text{and} \quad 3x - 2 < 1$$

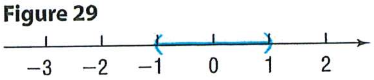
We solve each of these inequalities separately.

$-5 < 3x - 2$		$3x - 2 < 1$
$-5 + 2 < 3x - 2 + 2$	Add 2 to both sides.	$3x - 2 + 2 < 1 + 2$
$-3 < 3x$	Simplify.	$3x < 3$
$\frac{-3}{3} < \frac{3x}{3}$	Divide both sides by 3.	$\frac{3x}{3} < \frac{3}{3}$
$-1 < x$	Simplify.	$x < 1$

The solution set of the original pair of inequalities consists of all x for which

$$-1 < x \quad \text{and} \quad x < 1$$

Figure 29



This may be written more compactly as $\{x|-1 < x < 1\}$. In interval notation, the solution is $(-1, 1)$. See Figure 29 for the graph.

Observe in the preceding process that the two inequalities we solved required exactly the same steps. A shortcut to solving the original inequality algebraically is to deal with the two inequalities at the same time, as follows:

$$-5 < 3x - 2 < 1$$
$$-5 + 2 < 3x - 2 + 2 < 1 + 2 \quad \text{Add 2 to each part.}$$
$$-3 < 3x < 3 \quad \text{Simplify.}$$
$$\frac{-3}{3} < \frac{3x}{3} < \frac{3}{3} \quad \text{Divide each part by 3.}$$
$$-1 < x < 1 \quad \text{Simplify.}$$

Now Work PROBLEM 73

EXAMPLE 6 Using the Reciprocal Property to Solve an Inequality

Solve the inequality: $(4x - 1)^{-1} > 0$
Graph the solution set.

Solution Since $(4x - 1)^{-1} = \dfrac{1}{4x - 1}$ and since the Reciprocal Property states that when $\dfrac{1}{a} > 0$ then $a > 0$, we have

$$\begin{aligned} (4x - 1)^{-1} &> 0 \\ \frac{1}{4x - 1} &> 0 \\ 4x - 1 &> 0 \quad \text{Reciprocal Property} \\ 4x &> 1 \\ x &> \frac{1}{4} \end{aligned}$$

Figure 30

The solution set is $\left\{x \middle| x > \dfrac{1}{4}\right\}$, that is, all x in the interval $\left(\dfrac{1}{4}, \infty\right)$. Figure 30 illustrates the graph.

Now Work PROBLEM 83

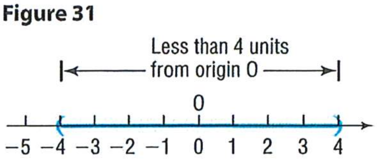
5 Solve Inequalities Involving Absolute Value

EXAMPLE 7 Solving an Inequality Involving Absolute Value

Solve the inequality $|x| < 4$, and graph the solution set.

Solution We are looking for all points whose coordinate x is a distance less than 4 units from the origin. See Figure 31 for an illustration. Because any x between -4 and 4 satisfies the condition $|x| < 4$, the solution set consists of all numbers x for which $-4 < x < 4$, that is, all x in the interval $(-4, 4)$.

Figure 31

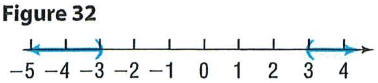


EXAMPLE 8 Solving an Inequality Involving Absolute Value

Solve the inequality $|x| > 3$, and graph the solution set.

Solution We are looking for all points whose coordinate x is a distance greater than 3 units from the origin. Figure 32 illustrates the situation. We conclude that any x less than -3 or greater than 3 satisfies the condition $|x| > 3$. The solution set consists of all numbers x for which $x < -3$ or $x > 3$, that is, all x in $(-\infty, -3) \cup (3, \infty)$.*

Figure 32



We are led to the following results:

THEOREM

If a is any positive number, then

$\lvert u\rvert < a$	is equivalent to	$-a < u < a$	**(5)**
$\lvert u\rvert \le a$	is equivalent to	$-a \le u \le a$	**(6)**
$\lvert u\rvert > a$	is equivalent to	$u < -a$ or $u > a$	**(7)**
$\lvert u\rvert \ge a$	is equivalent to	$u \le -a$ or $u \ge a$	**(8)**

*The symbol $\cup$ stands for the union of two sets. Refer to page A2 if necessary.

EXAMPLE 9 **Solving an Inequality Involving Absolute Value**

Solve the inequality $|2x + 4| \leq 3$, and graph the solution set.

Solution

$$|2x + 4| \leq 3$$ This follows the form of statement (6); the expression $u = 2x + 4$ is inside the absolute value bars.

$$-3 \leq 2x + 4 \leq 3$$ Apply statement (6).

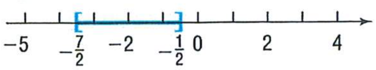
$$-3 - 4 \leq 2x + 4 - 4 \leq 3 - 4$$ Subtract 4 from each part.

$$-7 \leq 2x \leq -1$$ Simplify.

$$\frac{-7}{2} \leq \frac{2x}{2} \leq \frac{-1}{2}$$ Divide each part by 2.

$$-\frac{7}{2} \leq x \leq -\frac{1}{2}$$ Simplify.

Figure 33



The solution set is $\left\{x \middle| -\frac{7}{2} \leq x \leq -\frac{1}{2}\right\}$, that is, all x in the interval $\left[-\frac{7}{2}, -\frac{1}{2}\right]$. See Figure 33 for a graph of the solution set.

Now Work PROBLEM 89

EXAMPLE 10 **Solving an Inequality Involving Absolute Value**

Solve the inequality $|2x - 5| > 3$, and graph the solution set.

Solution

$$|2x - 5| > 3$$ This follows the form of statement (7); the expression $u = 2x - 5$ is inside the absolute value bars.

$$2x - 5 < -3 \quad \text{or} \quad 2x - 5 > 3$$ Apply statement (7).

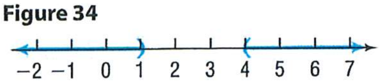
$$2x - 5 + 5 < -3 + 5 \quad \text{or} \quad 2x - 5 + 5 > 3 + 5$$ Add 5 to each part.

$$2x < 2 \quad \text{or} \quad 2x > 8$$ Simplify.

$$\frac{2x}{2} < \frac{2}{2} \quad \text{or} \quad \frac{2x}{2} > \frac{8}{2}$$ Divide each part by 2.

$$x < 1 \quad \text{or} \quad x > 4$$ Simplify.

Figure 34



The solution set is $\{x | x < 1 \text{ or } x > 4\}$, that is, all x in $(-\infty, 1) \cup (4, \infty)$. See Figure 34 for a graph of the solution set.

WARNING A common error to be avoided is to attempt to write the solution $x < 1$ or $x > 4$ as the combined inequality $1 > x > 4$, which is incorrect, since there are no numbers x for which $x < 1$ *and* $x > 4$. Another common error is to "mix" the symbols and write $1 < x > 4$, which makes no sense.

Now Work PROBLEM 95

A.9 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

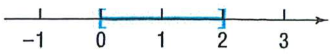
1. Graph the inequality: $x \geq -2$. (pp. A4–A5)
2. ***True or False*** $-5 > -3$ (pp. A4–A5)
3. $|-2| =$ ______. (p. A5)
4. ***True or False*** $|x| \geq 0$ for any real number x. (pp. A5–A6)

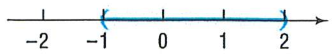
Concepts and Vocabulary

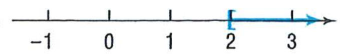
5. If each side of an inequality is multiplied by a(n) ______ number, then the sense of the inequality symbol is reversed.
6. A(n) ______ ______, denoted $[a, b]$, consists of all real numbers x for which $a \leq x \leq b$.
7. The solution set of the equation $|x| = 5$ is {______}.
8. The solution set of the inequality $|x| < 5$ is $\{x|$______$\}$
9. ***True or False*** The equation $|x| = -2$ has no solution.
10. ***True or False*** The inequality $|x| \geq -2$ has the set of real numbers as solution set.

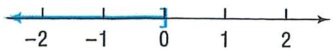
Skill Building

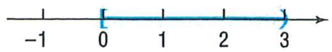
In Problems 11–16, express the graph shown in blue using interval notation. Also express each as an inequality involving x.

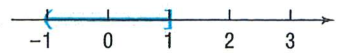
11. 

12. 

13. 

14. 

15. 

16. 

In Problems 17–22, an inequality is given. Write the inequality obtained by:
(a) Adding 3 to each side of the given inequality.
(b) Subtracting 5 from each side of the given inequality.
(c) Multiplying each side of the given inequality by 3.
(d) Multiplying each side of the given inequality by −2.

17. $3 < 5$ **18.** $2 > 1$ **19.** $4 > -3$ **20.** $-3 > -5$ **21.** $2x + 1 < 2$ **22.** $1 - 2x > 5$

In Problems 23–30, write each inequality using interval notation, and illustrate each inequality using the real number line.

23. $0 \le x \le 4$ **24.** $-1 < x < 5$ **25.** $4 \le x < 6$ **26.** $-2 < x < 0$

27. $x \ge 4$ **28.** $x \le 5$ **29.** $x < -4$ **30.** $x > 1$

In Problems 31–38, write each interval as an inequality involving x, and illustrate each inequality using the real number line.

31. $[2, 5]$ **32.** $(1, 2)$ **33.** $(-3, -2)$ **34.** $[0, 1)$

35. $[4, \infty)$ **36.** $(-\infty, 2]$ **37.** $(-\infty, -3)$ **38.** $(-8, \infty)$

In Problems 39–52, fill in the blank with the correct inequality symbol.

39. If $x < 5$, then $x - 5$ ________ 0.

40. If $x < -4$, then $x + 4$ ________ 0.

41. If $x > -4$, then $x + 4$ ________ 0.

42. If $x > 6$, then $x - 6$ ________ 0.

43. If $x \ge -4$, then $3x$ ________ -12.

44. If $x \le 3$, then $2x$ ________ 6.

45. If $x > 6$, then $-2x$ ________ -12.

46. If $x > -2$, then $-4x$ ________ 8.

47. If $x \ge 5$, then $-4x$ ________ -20.

48. If $x \le -4$, then $-3x$ ________ 12.

49. If $2x > 6$, then x ________ 3.

50. If $3x \le 12$, then x ________ 4.

51. If $-\frac{1}{2}x \le 3$, then x ________ -6.

52. If $-\frac{1}{4}x > 1$, then x ________ -4.

In Problems 53–100, solve each inequality. Express your answer using set notation or interval notation. Graph the solution set.

53. $x + 1 < 5$ **54.** $x - 6 < 1$ **55.** $1 - 2x \le 3$

56. $2 - 3x \le 5$ **57.** $3x - 7 > 2$ **58.** $2x + 5 > 1$

59. $3x - 1 \ge 3 + x$ **60.** $2x - 2 \ge 3 + x$ **61.** $-2(x + 3) < 8$

62. $-3(1 - x) < 12$ **63.** $4 - 3(1 - x) \le 3$ **64.** $8 - 4(2 - x) \le -2x$

65. $\frac{1}{2}(x - 4) > x + 8$ **66.** $3x + 4 > \frac{1}{3}(x - 2)$ **67.** $\frac{x}{2} \ge 1 - \frac{x}{4}$

68. $\frac{x}{3} \ge 2 + \frac{x}{6}$ **69.** $0 \le 2x - 6 \le 4$ **70.** $4 \le 2x + 2 \le 10$

71. $-5 \le 4 - 3x \le 2$ **72.** $-3 \le 3 - 2x \le 9$ **73.** $-3 < \frac{2x - 1}{4} < 0$

74. $0 < \frac{3x + 2}{2} < 4$ **75.** $1 < 1 - \frac{1}{2}x < 4$ **76.** $0 < 1 - \frac{1}{3}x < 1$

77. $(x + 2)(x - 3) > (x - 1)(x + 1)$

78. $(x - 1)(x + 1) > (x - 3)(x + 4)$

79. $x(4x + 3) \le (2x + 1)^2$

80. $x(9x - 5) \le (3x - 1)^2$

81. $\frac{1}{2} \le \frac{x+1}{3} < \frac{3}{4}$

82. $\frac{1}{3} < \frac{x+1}{2} \le \frac{2}{3}$

83. $(4x + 2)^{-1} < 0$

84. $(2x - 1)^{-1} > 0$

85. $0 < \frac{2}{x} < \frac{3}{5}$

86. $0 < \frac{4}{x} < \frac{2}{3}$

87. $0 < (2x - 4)^{-1} < \frac{1}{2}$

88. $0 < (3x + 6)^{-1} < \frac{1}{3}$

89. $|2x| < 8$

90. $|3x| < 12$

91. $|3x| > 12$

92. $|2x| > 6$

93. $|2x - 1| \le 1$

94. $|2x + 5| \le 7$

95. $|1 - 2x| > 3$

96. $|2 - 3x| > 1$

97. $|-4x| + |-5| \le 9$

98. $|-x| - |4| \le 2$

99. $|-2x| \ge |-4|$

100. $|-x - 2| \ge 1$

Applications and Extensions

101. Express the fact that x differs from 2 by less than $\frac{1}{2}$ as an inequality involving an absolute value. Solve for x.

102. Express the fact that x differs from -1 by less than 1 as an inequality involving an absolute value. Solve for x.

103. Express the fact that x differs from -3 by more than 2 as an inequality involving an absolute value. Solve for x.

104. Express the fact that x differs from 2 by more than 3 as an inequality involving an absolute value. Solve for x.

105. What is the domain of the variable in the expression $\sqrt{3x + 6}$?

106. What is the domain of the variable in the expression $\sqrt{8 + 2x}$?

107. A young adult may be defined as someone older than 21, but less than 30 years of age. Express this statement using inequalities.

108. Middle-aged may be defined as being 40 or more and less than 60. Express this statement using inequalities.

109. Life Expectancy The Social Security Administration determined that an average 30-year-old male in 2005 could expect to live at least 46.60 more years and an average 30-year-old female in 2005 could expect to live at least 51.03 more years.

(a) To what age can an average 30-year-old male expect to live? Express your answer as an inequality.

(b) To what age can an average 30-year-old female expect to live? Express your answer as an inequality.

(c) Who can expect to live longer, a male or a female? By how many years?

Source: *Social Security Administration, Period Life Table, 2005*

110. General Chemistry For a certain ideal gas, the volume V (in cubic centimeters) equals 20 times the temperature T (in degrees Celsius). If the temperature varies from 80° to 120° C inclusive, what is the corresponding range of the volume of the gas?

111. Real Estate A real estate agent agrees to sell an apartment complex according to the following commission schedule: \$45,000 plus 25% of the selling price in excess of \$900,000. Assuming that the complex will sell at some price between \$900,000 and \$1,100,000 inclusive, over what range does the agent's commission vary? How does the commission vary as a percent of selling price?

112. Sales Commission A used car salesperson is paid a commission of \$25 plus 40% of the selling price in excess of owner's cost. The owner claims that used cars typically sell for at least owner's cost plus \$200 and at most owner's cost plus \$3000. For each sale made, over what range can the salesperson expect the commission to vary?

113 Federal Tax Withholding The percentage method of withholding for federal income tax (2010) states that a single person whose weekly wages, after subtracting withholding allowances, are over \$693, but not over \$1302, shall have \$82.35 plus 25% of the excess over \$693 withheld. Over what range does the amount withheld vary if the weekly wages vary from \$700 to \$900 inclusive?

Source: Employer's Tax Guide. Department of the Treasury, Internal Revenue Service, Publication 2010.

114. Exercising Sue wants to lose weight. For healthy weight loss, the American College of Sports Medicine (ACSM) recommends 200 to 300 minutes of exercise per week. For the first six days of the week, Sue exercised 40, 45, 0, 50, 25, and 35 minutes. How long should Sue exercise on the seventh day in order to stay within the ACSM guidelines?

115. Electricity Rates Commonwealth Edison Company's charge for electricity in January 2010 is 9.44¢ per kilowatt-hour. In addition, each monthly bill contains a customer charge of \$12.55. If last year's bills ranged from a low of \$76.27 to a high of \$248.55, over what range did usage vary (in kilowatt-hours)?

Source: Commonwealth Edison Co., Chicago, Illinois, 2010.

116. Water Bills The Village of Oak Lawn charges homeowners \$37.62 per quarter-year plus \$3.86 per 1000 gallons for water usage in excess of 10,000 gallons. In 2010 one homeowner's quarterly bill ranged from a high of \$122.54 to a low of \$68.50. Over what range did water usage vary?

Source: Village of Oak Lawn, Illinois, April 2010.

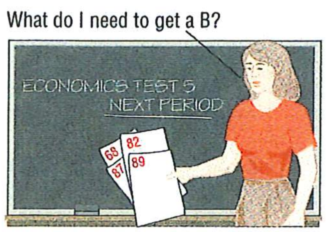
117. Markup of a New Car The markup over dealer's cost of a new car ranges from 12% to 18%. If the sticker price is \$18,000, over what range will the dealer's cost vary?

118. IQ Tests A standard intelligence test has an average score of 100. According to statistical theory, of the people who take the test, the 2.5% with the highest scores will have scores of more than 1.96σ above the average, where σ (sigma, a number called the **standard deviation**) depends on the nature of the test. If $\sigma = 12$ for this test and there is (in principle) no upper limit to the score possible on the test, write the interval of possible test scores of the people in the top 2.5%.

119. Computing Grades In your Economics 101 class, you have scores of 68, 82, 87, and 89 on the first four of five tests. To get a grade of B, the average of the first five test scores must be greater than or equal to 80 and less than 90.

(a) Solve an inequality to find the range of the score that you need on the last test to get a B.

(b) What score do you need if the fifth test counts double?



120. "Light" Foods For food products to be labeled "light," the U.S. Food and Drug Administration requires that the altered product must either contain one-third or fewer calories than the regular product or it must contain one-half or less fat than the regular product. If a serving of Miracle Whip® Light contains 20 calories and 1.5 grams of fat, then what must be true about either the number of calories or the grams of fat in a serving of regular Miracle Whip®?

121. Arithmetic Mean If $a < b$, show that $a < \frac{a+b}{2} < b$. The number $\frac{a+b}{2}$ is called the **arithmetic mean** of a and b.

122. Refer to Problem 121. Show that the arithmetic mean of a and b is equidistant from a and b.

123. Geometric Mean If $0 < a < b$, show that $a < \sqrt{ab} < b$. The number $\sqrt{ab}$ is called the **geometric mean** of a and b.

124. Refer to Problems 121 and 123. Show that the geometric mean of a and b is less than the arithmetic mean of a and b.

125. Harmonic Mean For $0 < a < b$, let h be defined by

$$\frac{1}{h} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Show that $a < h < b$. The number h is called the **harmonic mean** of a and b.

126. Refer to Problems 121, 123, and 125. Show that the harmonic mean of a and b equals the geometric mean squared divided by the arithmetic mean.

127. Another Reciprocal Property Prove that if $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

Explaining Concepts: Discussion and Writing

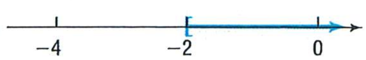
128. Make up an inequality that has no solution. Make up one that has exactly one solution.

129. The inequality $x^2 + 1 < -5$ has no real solution. Explain why.

130. Do you prefer to use inequality notation or interval notation to express the solution to an inequality? Give your reasons. Are there particular circumstances when you prefer one to the other? Cite examples.

131. How would you explain to a fellow student the underlying reason for the multiplication properties for inequalities (page A74), that is, the sense or direction of an inequality remains the same if each side is multiplied by a positive real number, whereas the direction is reversed if each side is multiplied by a negative real number?

'Are You Prepared?' Answers

1. 

2. False **3.** 2 **4.** True

A.10 *n*th Roots; Rational Exponents

PREPARING FOR THIS SECTION *Before getting started, review the following:*

- Exponents, Square Roots (Appendix A, Section A.1, pp. A7–A10)

Now Work the **'Are You Prepared?'** problems on page A87.

OBJECTIVES
1 Work with *n*th Roots (p. A82)
2 Simplify Radicals (p. A82)
3 Rationalize Denominators (p. A84)
4 Solve Radical Equations (p. A84)
5 Simplify Expressions with Rational Exponents (p. A85)

1 Work with *n*th Roots

DEFINITION

The **principal *n*th root of a real number *a*,** $n \geq 2$ an integer, symbolized by $\sqrt[n]{a}$, is defined as follows:

$$\sqrt[n]{a} = b \quad \text{means} \quad a = b^n$$

where $a \geq 0$ and $b \geq 0$ if n is even and a, b are any real numbers if n is odd.

> **In Words**
> The symbol $\sqrt[n]{a}$ means "give me the number, which when raised to the power n, equals a."

Notice that if a is negative and n is even, then $\sqrt[n]{a}$ is not defined as a real number. When it is defined, the principal nth root of a number is unique.

The symbol $\sqrt[n]{a}$ for the principal nth root of a is called a **radical;** the integer n is called the **index,** and a is called the **radicand.** If the index of a radical is 2, we call $\sqrt[2]{a}$ the **square root** of a and omit the index 2 by simply writing $\sqrt{a}$. If the index is 3, we call $\sqrt[3]{a}$ the **cube root** of a.

EXAMPLE 1 **Simplifying Principal *n*th Roots**

(a) $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$

(b) $\sqrt[3]{-64} = \sqrt[3]{(-4)^3} = -4$

(c) $\sqrt[4]{\dfrac{1}{16}} = \sqrt[4]{\left(\dfrac{1}{2}\right)^4} = \dfrac{1}{2}$

(d) $\sqrt[6]{(-2)^6} = |-2| = 2$

These are examples of **perfect roots,** since each simplifies to a rational number. Notice the absolute value in Example 1(d). If n is even, the principal nth root must be nonnegative.

In general, if $n \geq 2$ is an integer and a is a real number, we have

$$\sqrt[n]{a^n} = a \quad \text{if } n \geq 3 \text{ is odd} \qquad \textbf{(1a)}$$

$$\sqrt[n]{a^n} = |a| \quad \text{if } n \geq 2 \text{ is even} \qquad \textbf{(1b)}$$

Now Work PROBLEM 7

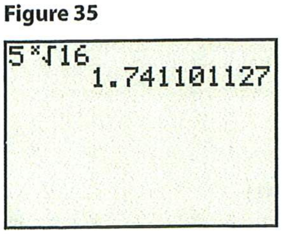
Radicals provide a way of representing many irrational real numbers. For example, there is no rational number whose square is 2. Using radicals, we can say that $\sqrt{2}$ is the positive number whose square is 2.

EXAMPLE 2 **Using a Calculator to Approximate Roots**

Use a calculator to approximate $\sqrt[5]{16}$.

Solution Figure 35 shows the result using a TI-84 Plus graphing calculator.

Figure 35



Now Work PROBLEM 81

2 Simplify Radicals

Let $n \geq 2$ and $m \geq 2$ denote positive integers, and let a and b represent real numbers. Assuming that all radicals are defined, we have the following properties:

Properties of Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a}\,\sqrt[n]{b} \qquad \textbf{(2a)}$$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \textbf{(2b)}$$

$$\sqrt[n]{a^m} = (\sqrt[n]{a})^m \qquad \textbf{(2c)}$$

When used in reference to radicals, the direction to "simplify" will mean to remove from the radicals any perfect roots that occur as factors. Let's look at some examples of how the preceding rules are applied to simplify radicals.

EXAMPLE 3 **Simplifying Radicals**

(a) $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

Factor out 16, a perfect square. (2a)

(b) $\sqrt[3]{16} = \sqrt[3]{8 \cdot 2} = \sqrt[3]{8} \cdot \sqrt[3]{2} = \sqrt[3]{2^3} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}$

Factor out 8, a perfect cube. (2a)

(c) $\sqrt[3]{-16x^4} = \sqrt[3]{-8 \cdot 2 \cdot x^3 \cdot x} = \sqrt[3]{(-8x^3)(2x)}$

Factor perfect cubes inside radical. Group perfect cubes.

$= \sqrt[3]{(-2x)^3 \cdot 2x} = \sqrt[3]{(-2x)^3} \cdot \sqrt[3]{2x} = -2x\sqrt[3]{2x}$

(2a)

(d) $\sqrt[4]{\frac{16x^5}{81}} = \sqrt[4]{\frac{2^4x^4x}{3^4}} = \sqrt[4]{\left(\frac{2x}{3}\right)^4 \cdot x} = \sqrt[4]{\left(\frac{2x}{3}\right)^4} \cdot \sqrt[4]{x} = \left|\frac{2x}{3}\right| \sqrt[4]{x}$

Now Work PROBLEMS 11 AND 17

Two or more radicals can be combined, provided that they have the same index and the same radicand. Such radicals are called **like radicals.**

EXAMPLE 4 **Combining Like Radicals**

(a) $-8\sqrt{12} + \sqrt{3} = -8\sqrt{4 \cdot 3} + \sqrt{3}$
$= -8 \cdot \sqrt{4}\sqrt{3} + \sqrt{3}$
$= -16\sqrt{3} + \sqrt{3} = -15\sqrt{3}$

(b) $\sqrt[3]{8x^4} + \sqrt[3]{-x} + 4\sqrt[3]{27x} = \sqrt[3]{2^3x^3x} + \sqrt[3]{-1 \cdot x} + 4\sqrt[3]{3^3x}$
$= \sqrt[3]{(2x)^3} \cdot \sqrt[3]{x} + \sqrt[3]{-1} \cdot \sqrt[3]{x} + 4\sqrt[3]{3^3} \cdot \sqrt[3]{x}$
$= 2x\sqrt[3]{x} - 1 \cdot \sqrt[3]{x} + 12\sqrt[3]{x}$
$= (2x + 11)\sqrt[3]{x}$

Now Work PROBLEM 33

3 Rationalize Denominators

When radicals occur in the denominator of a quotient, it is customary to rewrite the quotient so that the new denominator contains no radicals. This process is referred to as **rationalizing the denominator.**

The idea is to multiply by an appropriate expression so that the new denominator contains no radicals. For example:

If a Denominator Contains the Factor	Multiply by	To Obtain a Denominator Free of Radicals
$\sqrt{3}$	$\sqrt{3}$	$(\sqrt{3})^2 = 3$
$\sqrt{3} + 1$	$\sqrt{3} - 1$	$(\sqrt{3})^2 - 1^2 = 3 - 1 = 2$
$\sqrt{2} - 3$	$\sqrt{2} + 3$	$(\sqrt{2})^2 - 3^2 = 2 - 9 = -7$
$\sqrt{5} - \sqrt{3}$	$\sqrt{5} + \sqrt{3}$	$(\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3 = 2$
$\sqrt[3]{4}$	$\sqrt[3]{2}$	$\sqrt[3]{4} \cdot \sqrt[3]{2} = \sqrt[3]{8} = 2$

In rationalizing the denominator of a quotient, be sure to multiply both the numerator and the denominator by the expression.

EXAMPLE 5 **Rationalizing Denominators**

Rationalize the denominator of each expression.

(a) $\dfrac{4}{\sqrt{2}}$ (b) $\dfrac{\sqrt{3}}{\sqrt[3]{2}}$ (c) $\dfrac{\sqrt{x} - 2}{\sqrt{x} + 2}, \quad x \ge 0$

Solution

(a) $$\frac{4}{\sqrt{2}} = \frac{4}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{4\sqrt{2}}{(\sqrt{2})^2} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}$$

Multiply by $\dfrac{\sqrt{2}}{\sqrt{2}}$.

(b) $$\frac{\sqrt{3}}{\sqrt[3]{2}} = \frac{\sqrt{3}}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \frac{\sqrt{3}\sqrt[3]{4}}{\sqrt[3]{8}} = \frac{\sqrt{3}\sqrt[3]{4}}{2}$$

Multiply by $\dfrac{\sqrt[3]{4}}{\sqrt[3]{4}}$.

(c) $$\frac{\sqrt{x} - 2}{\sqrt{x} + 2} = \frac{\sqrt{x} - 2}{\sqrt{x} + 2} \cdot \frac{\sqrt{x} - 2}{\sqrt{x} - 2} = \frac{(\sqrt{x} - 2)^2}{(\sqrt{x})^2 - 2^2}$$

$$= \frac{(\sqrt{x})^2 - 4\sqrt{x} + 4}{x - 4} = \frac{x - 4\sqrt{x} + 4}{x - 4}$$

Now Work PROBLEM 47

4 Solve Radical Equations

When the variable in an equation occurs in a square root, cube root, and so on, that is, when it occurs under a radical, the equation is called a **radical equation.** Sometimes a suitable operation will change a radical equation to one that is linear or quadratic. The most commonly used procedure is to isolate the most complicated radical on one side of the equation and then eliminate it by raising each side to a power equal to the index of the radical. Care must be taken because extraneous solutions may result. Thus, when working with radical equations, we always check apparent solutions. Let's look at an example.

EXAMPLE 6 **Solving Radical Equations**

Solve the equation: $\sqrt[3]{2x-4} - 2 = 0$

Solution The equation contains a radical whose index is 3. We isolate it on the left side.

$$\sqrt[3]{2x-4} - 2 = 0$$
$$\sqrt[3]{2x-4} = 2$$

Now raise each side to the third power (since the index of the radical is 3) and solve.

$$\left(\sqrt[3]{2x-4}\right)^3 = 2^3 \quad \text{Raise each side to the third power.}$$
$$2x - 4 = 8 \quad \text{Simplify.}$$
$$2x = 12 \quad \text{Simplify.}$$
$$x = 6 \quad \text{Solve for } x.$$

Check: $\sqrt[3]{2(6)-4} - 2 = \sqrt[3]{12-4} - 2 = \sqrt[3]{8} - 2 = 2 - 2 = 0.$

The solution is $x = 6$.

Now Work PROBLEM 55

5 Simplify Expressions with Rational Exponents

Radicals are used to define rational exponents.

DEFINITION

If a is a real number and $n \geq 2$ is an integer, then

$$a^{1/n} = \sqrt[n]{a} \quad (3)$$

provided that $\sqrt[n]{a}$ exists.

Note that if n is even and $a < 0$, then $\sqrt[n]{a}$ and $a^{1/n}$ do not exist as real numbers.

EXAMPLE 7 **Writing Expressions Containing Fractional Exponents as Radicals**

(a) $4^{1/2} = \sqrt{4} = 2$

(b) $8^{1/2} = \sqrt{8} = 2\sqrt{2}$

(c) $(-27)^{1/3} = \sqrt[3]{-27} = -3$

(d) $16^{1/3} = \sqrt[3]{16} = 2\sqrt[3]{2}$

DEFINITION

If a is a real number and m and n are integers containing no common factors, with $n \geq 2$, then

$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \quad (4)$$

provided that $\sqrt[n]{a}$ exists.

We have two comments about equation (4):

1. The exponent $\frac{m}{n}$ must be in lowest terms and n must be positive.
2. In simplifying the rational expression $a^{m/n}$, either $\sqrt[n]{a^m}$ or $\left(\sqrt[n]{a}\right)^m$ may be used, the choice depending on which is easier to simplify. Generally, taking the root first, as in $\left(\sqrt[n]{a}\right)^m$, is easier.

EXAMPLE 8 **Using Equation (4)**

(a) $4^{3/2} = (\sqrt{4})^3 = 2^3 = 8$

(b) $(-8)^{4/3} = (\sqrt[3]{-8})^4 = (-2)^4 = 16$

(c) $(32)^{-2/5} = (\sqrt[5]{32})^{-2} = 2^{-2} = \frac{1}{4}$

(d) $25^{6/4} = 25^{3/2} = (\sqrt{25})^3 = 5^3 = 125$

Now Work PROBLEM 59

It can be shown that the Laws of Exponents hold for rational exponents. The next example illustrates using the Laws of Exponents to simplify.

EXAMPLE 9 **Simplifying Expressions Containing Rational Exponents**

Simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

(a) $(x^{2/3}y)(x^{-2}y)^{1/2}$ (b) $\left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3}$ (c) $\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2}$

Solution

(a)
$$
\begin{aligned}
(x^{2/3}y)(x^{-2}y)^{1/2} &= (x^{2/3}y)[(x^{-2})^{1/2}y^{1/2}] \\
&= x^{2/3}yx^{-1}y^{1/2} \\
&= (x^{2/3}\cdot x^{-1})(y\cdot y^{1/2}) \\
&= x^{-1/3}y^{3/2} \\
&= \frac{y^{3/2}}{x^{1/3}}
\end{aligned}
$$

(b)
$$\left(\frac{2x^{1/3}}{y^{2/3}}\right)^{-3} = \left(\frac{y^{2/3}}{2x^{1/3}}\right)^{3} = \frac{(y^{2/3})^3}{(2x^{1/3})^3} = \frac{y^2}{2^3(x^{1/3})^3} = \frac{y^2}{8x}$$

(c)
$$\left(\frac{9x^2y^{1/3}}{x^{1/3}y}\right)^{1/2} = \left(\frac{9x^{2-(1/3)}}{y^{1-(1/3)}}\right)^{1/2} = \left(\frac{9x^{5/3}}{y^{2/3}}\right)^{1/2} = \frac{9^{1/2}(x^{5/3})^{1/2}}{(y^{2/3})^{1/2}} = \frac{3x^{5/6}}{y^{1/3}}$$

Now Work PROBLEM 75

The next two examples illustrate some algebra that you will need to know for certain calculus problems.

EXAMPLE 10 **Writing an Expression as a Single Quotient**

Write the following expression as a single quotient in which only positive exponents appear.

$$(x^2+1)^{1/2} + x\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x$$

Solution

$$
\begin{aligned}
(x^2+1)^{1/2} + x\cdot\frac{1}{2}(x^2+1)^{-1/2}\cdot 2x &= (x^2+1)^{1/2} + \frac{x^2}{(x^2+1)^{1/2}} \\
&= \frac{(x^2+1)^{1/2}(x^2+1)^{1/2} + x^2}{(x^2+1)^{1/2}} \\
&= \frac{(x^2+1) + x^2}{(x^2+1)^{1/2}} \\
&= \frac{2x^2+1}{(x^2+1)^{1/2}}
\end{aligned}
$$

Now Work PROBLEM 89

EXAMPLE 11 **Factoring an Expression Containing Rational Exponents**

Factor: $\frac{4}{3}x^{1/3}(2x + 1) + 2x^{4/3}$

Solution We begin by writing $2x^{4/3}$ as a fraction with 3 as denominator.

$$\frac{4}{3}x^{1/3}(2x + 1) + 2x^{4/3} = \frac{4x^{1/3}(2x + 1)}{3} + \frac{6x^{4/3}}{3} = \frac{4x^{1/3}(2x + 1) + 6x^{4/3}}{3}$$

Add the two fractions.

$$= \frac{2x^{1/3}[2(2x + 1) + 3x]}{3} = \frac{2x^{1/3}(7x + 2)}{3}$$

2 and $x^{1/3}$ are common factors. Simplify.

Now Work PROBLEM 101

A.10 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. $(-3)^2 =$ ______; $-3^2 =$ ______ (pp. A7–A9)

2. $\sqrt{16} =$ ______; $\sqrt{(-4)^2} =$ ______ (pp. A9–A10)

Concepts and Vocabulary

3. In the symbol $\sqrt[n]{a}$, the integer n is called the ______.

4. ***True or False*** $\sqrt[5]{-32} = -2$

5. We call $\sqrt[3]{a}$ the ______ ______ of a.

6. ***True or False*** $\sqrt[4]{(-3)^4} = -3$

Skill Building

In Problems 7–42, simplify each expression. Assume that all variables are positive when they appear.

7. $\sqrt[3]{27}$ **8.** $\sqrt[4]{16}$ **9.** $\sqrt[3]{-8}$ **10.** $\sqrt[3]{-1}$

11. $\sqrt{8}$ **12.** $\sqrt[3]{54}$ **13.** $\sqrt[3]{-8x^4}$ **14.** $\sqrt[4]{48x^5}$

15. $\sqrt[4]{x^{12}y^8}$ **16.** $\sqrt[5]{x^{10}y^5}$ **17.** $\sqrt[4]{\frac{x^9y^7}{xy^3}}$ **18.** $\sqrt[3]{\frac{3xy^2}{81x^4y^2}}$

19. $\sqrt{36x}$ **20.** $\sqrt{9x^5}$ **21.** $\sqrt{3x^2}\sqrt{12x}$ **22.** $\sqrt{5x}\sqrt{20x^3}$

23. $(\sqrt{5}\sqrt[3]{9})^2$ **24.** $(\sqrt[3]{3}\sqrt{10})^4$ **25.** $(3\sqrt{6})(2\sqrt{2})$ **26.** $(5\sqrt{8})(-3\sqrt{3})$

27. $3\sqrt{2} + 4\sqrt{2}$ **28.** $6\sqrt{5} - 4\sqrt{5}$ **29.** $-\sqrt{18} + 2\sqrt{8}$ **30.** $2\sqrt{12} - 3\sqrt{27}$

31. $(\sqrt{3} + 3)(\sqrt{3} - 1)$ **32.** $(\sqrt{5} - 2)(\sqrt{5} + 3)$ **33.** $5\sqrt[3]{2} - 2\sqrt[3]{54}$ **34.** $9\sqrt[3]{24} - \sqrt[3]{81}$

35. $(\sqrt{x} - 1)^2$ **36.** $(\sqrt{x} + \sqrt{5})^2$ **37.** $\sqrt[3]{16x^4} - \sqrt[3]{2x}$ **38.** $\sqrt[4]{32x} + \sqrt[4]{2x^5}$

39. $\sqrt{8x^3} - 3\sqrt{50x}$ **40.** $3x\sqrt{9y} + 4\sqrt{25y}$

41. $\sqrt[3]{16x^4y} - 3x\sqrt[3]{2xy} + 5\sqrt[3]{-2xy^4}$ **42.** $8xy - \sqrt{25x^2y^2} + \sqrt[3]{8x^3y^3}$

In Problems 43–54, rationalize the denominator of each expression. Assume that all variables are positive when they appear.

43. $\dfrac{1}{\sqrt{2}}$ **44.** $\dfrac{2}{\sqrt{3}}$ **45.** $\dfrac{-\sqrt{3}}{\sqrt{5}}$ **46.** $\dfrac{-\sqrt{3}}{\sqrt{8}}$

47. $\dfrac{\sqrt{3}}{5-\sqrt{2}}$ **48.** $\dfrac{\sqrt{2}}{\sqrt{7}+2}$ **49.** $\dfrac{2-\sqrt{5}}{2+3\sqrt{5}}$ **50.** $\dfrac{\sqrt{3}-1}{2\sqrt{3}+3}$

51. $\dfrac{5}{\sqrt[3]{2}}$ **52.** $\dfrac{-2}{\sqrt[3]{9}}$ **53.** $\dfrac{\sqrt{x+h}-\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}$ **54.** $\dfrac{\sqrt{x+h}+\sqrt{x-h}}{\sqrt{x+h}-\sqrt{x-h}}$

In Problems 55–58, solve each equation.

55. $\sqrt[3]{2t-1}=2$ **56.** $\sqrt[3]{3t+1}=-2$ **57.** $\sqrt{15-2x}=x$ **58.** $\sqrt{12-x}=x$

In Problems 59–70, simplify each expression.

59. $8^{2/3}$ **60.** $4^{3/2}$ **61.** $(-27)^{1/3}$ **62.** $16^{3/4}$ **63.** $16^{3/2}$ **64.** $25^{3/2}$

65. $9^{-3/2}$ **66.** $16^{-3/2}$ **67.** $\left(\dfrac{9}{8}\right)^{3/2}$ **68.** $\left(\dfrac{27}{8}\right)^{2/3}$ **69.** $\left(\dfrac{8}{9}\right)^{-3/2}$ **70.** $\left(\dfrac{8}{27}\right)^{-2/3}$

In Problems 71–78, simplify each expression. Express your answer so that only positive exponents occur. Assume that the variables are positive.

71. $x^{3/4}x^{1/3}x^{-1/2}$ **72.** $x^{2/3}x^{1/2}x^{-1/4}$ **73.** $(x^3y^6)^{1/3}$ **74.** $(x^4y^8)^{3/4}$

75. $\dfrac{(x^2y)^{1/3}(xy^2)^{2/3}}{x^{2/3}y^{2/3}}$ **76.** $\dfrac{(xy)^{1/4}(x^2y^2)^{1/2}}{(x^2y)^{3/4}}$ **77.** $\dfrac{(16x^2y^{-1/3})^{3/4}}{(xy^2)^{1/4}}$ **78.** $\dfrac{(4x^{-1}y^{1/3})^{3/2}}{(xy)^{3/2}}$

In Problems 79–86, use a calculator to approximate each radical. Round your answer to two decimal places.

79. $\sqrt{2}$ **80.** $\sqrt{7}$ **81.** $\sqrt[3]{4}$ **82.** $\sqrt[3]{-5}$

83. $\dfrac{2+\sqrt{3}}{3-\sqrt{5}}$ **84.** $\dfrac{\sqrt{5}-2}{\sqrt{2}+4}$ **85.** $\dfrac{3\sqrt[3]{5}-\sqrt{2}}{\sqrt{3}}$ **86.** $\dfrac{2\sqrt{3}-\sqrt[3]{4}}{\sqrt{2}}$

Applications and Extensions

In Problems 87–100, expressions that occur in calculus are given. Write each expression as a single quotient in which only positive exponents and/or radicals appear.

87. $\dfrac{x}{(1+x)^{1/2}}+2(1+x)^{1/2} \quad x>-1$

88. $\dfrac{1+x}{2x^{1/2}}+x^{1/2} \quad x>0$

89. $2x(x^2+1)^{1/2}+x^2\cdot\dfrac{1}{2}(x^2+1)^{-1/2}\cdot 2x$

90. $(x+1)^{1/3}+x\cdot\dfrac{1}{3}(x+1)^{-2/3} \quad x\neq -1$

91. $\sqrt{4x+3}\cdot\dfrac{1}{2\sqrt{x-5}}+\sqrt{x-5}\cdot\dfrac{1}{5\sqrt{4x+3}} \quad x>5$

92. $\dfrac{\sqrt[3]{8x+1}}{3\sqrt[3]{(x-2)^2}}+\dfrac{\sqrt[3]{x-2}}{24\sqrt[3]{(8x+1)^2}} \quad x\neq 2, x\neq -\dfrac{1}{8}$

93. $\dfrac{\sqrt{1+x}-x\cdot\dfrac{1}{2\sqrt{1+x}}}{1+x} \quad x>-1$

94. $\dfrac{\sqrt{x^2+1}-x\cdot\dfrac{2x}{2\sqrt{x^2+1}}}{x^2+1}$

95. $\dfrac{(x+4)^{1/2} - 2x(x+4)^{-1/2}}{x+4}$ $x > -4$

96. $\dfrac{(9-x^2)^{1/2} + x^2(9-x^2)^{-1/2}}{9-x^2}$ $-3 < x < 3$

97. $\dfrac{\dfrac{x^2}{(x^2-1)^{1/2}} - (x^2-1)^{1/2}}{x^2}$ $x < -1$ or $x > 1$

98. $\dfrac{(x^2+4)^{1/2} - x^2(x^2+4)^{-1/2}}{x^2+4}$

99. $\dfrac{\dfrac{1+x^2}{2\sqrt{x}} - 2x\sqrt{x}}{(1+x^2)^2}$ $x > 0$

100. $\dfrac{2x(1-x^2)^{1/3} + \dfrac{2}{3}x^3(1-x^2)^{-2/3}}{(1-x^2)^{2/3}}$ $x \neq -1, x \neq 1$

In Problems 101–110, expressions that occur in calculus are given. Factor each expression. Express your answer so that only positive exponents occur.

101. $(x+1)^{3/2} + x \cdot \frac{3}{2}(x+1)^{1/2}$ $x \geq -1$

102. $(x^2+4)^{4/3} + x \cdot \frac{4}{3}(x^2+4)^{1/3} \cdot 2x$

103. $6x^{1/2}(x^2+x) - 8x^{3/2} - 8x^{1/2}$ $x \geq 0$

104. $6x^{1/2}(2x+3) + x^{3/2} \cdot 8$ $x \geq 0$

105. $3(x^2+4)^{4/3} + x \cdot 4(x^2+4)^{1/3} \cdot 2x$

106. $2x(3x+4)^{4/3} + x^2 \cdot 4(3x+4)^{1/3}$

107. $4(3x+5)^{1/3}(2x+3)^{3/2} + 3(3x+5)^{4/3}(2x+3)^{1/2}$ $x \geq -\frac{3}{2}$

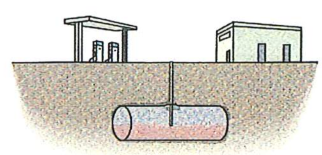
108. $6(6x+1)^{1/3}(4x-3)^{3/2} + 6(6x+1)^{4/3}(4x-3)^{1/2}$ $x \geq \frac{3}{4}$

109. $3x^{-1/2} + \frac{3}{2}x^{1/2}$ $x > 0$

110. $8x^{1/3} - 4x^{-2/3}$ $x \neq 0$

111. **Calculating the Amount of Gasoline in a Tank** A Shell station stores its gasoline in underground tanks that are right circular cylinders lying on their sides. See the illustration. The volume V of gasoline in the tank (in gallons) is given by the formula

$$V = 40h^2\sqrt{\frac{96}{h} - 0.608}$$

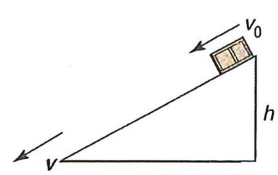


where h is the height of the gasoline (in inches) as measured on a depth stick.
(a) If $h = 12$ inches, how many gallons of gasoline are in the tank?
(b) If $h = 1$ inch, how many gallons of gasoline are in the tank?

112. **Inclined Planes** The final velocity v of an object in feet per second (ft/sec) after it slides down a frictionless inclined plane of height h feet is

$$v = \sqrt{64h + v_0^2}$$

where v_0 is the initial velocity (in ft/sec) of the object.
(a) What is the final velocity v of an object that slides down a frictionless inclined plane of height 4 feet? Assume that the initial velocity is 0.
(b) What is the final velocity v of an object that slides down a frictionless inclined plane of height 16 feet? Assume that the initial velocity is 0.
(c) What is the final velocity v of an object that slides down a frictionless inclined plane of height 2 feet with an initial velocity of 4 ft/sec?



Problems 113–116 require the following information.

Period of a Pendulum *The period T, in seconds, of a pendulum of length l, in feet, may be approximated using the formula*

$$T = 2\pi\sqrt{\frac{l}{32}}$$

In Problems 113–116, express your answer both as a square root and as a decimal.

113. Find the period T of a pendulum whose length is 64 feet.

114. Find the period T of a pendulum whose length is 16 feet.

115. Find the period T of a pendulum whose length is 8 inches.

116. Find the period T of a pendulum whose length is 4 inches.

Explaining Concepts: Discussion and Writing

117. Give an example to show that $\sqrt{a^2}$ is not equal to a. Use it to explain why $\sqrt{a^2} = |a|$.

'Are you Prepared?' Answers

1. 9; −9 **2.** 4; 4